9 Algebra Appendix to "Michelson-Morley, Fisher, and Occam: The Radical Implications of Stable Quiet Inflation at the Zero Bound."

9.1 Formulas for delayed and temporary rate rises

Here I work out the algebra for impulse response functions of the fiscal theory with long term debt model, with an announcement M years ahead of the interest rate rise, and an interest rate rise that only lasts M years, in both continuous and discrete time, Equations (32)-(33).

An interest rate rise from i to i^* that only lasts M years, continuous time:

$$\left[\vartheta\int_0^M e^{-i^*j}e^{-\vartheta j}dj+\vartheta\int_M^\infty e^{-i^*M-i(j-M)}e^{-\vartheta j}dj\right]\frac{B_t}{P_t^*}=\frac{s}{r}$$

$$\vartheta\left[\frac{e^{-(i^*+\vartheta)M}}{i+\vartheta}+\frac{1-e^{-(i^*+\vartheta)M}}{i^*+\vartheta}\right]\frac{B_t}{P_t^*}=\frac{s}{r}$$

$$\frac{P_t^*}{P_t}=\left(\frac{e^{-(i^*+\vartheta)M}}{i+\vartheta}+\frac{1-e^{-(i^*+\vartheta)M}}{i^*+\vartheta}\right)/\left(\frac{1}{i+\vartheta}\right)$$

$$\frac{P_t^*}{P_t}-1=\left(1-e^{-(i^*+\vartheta)M}\right)\left(\frac{i+\vartheta}{i^*+\vartheta}-1\right)\approx\left(1-e^{-\vartheta M}\right)\left(\frac{i+\vartheta}{i^*+\vartheta}-1\right)$$

An announcement of an interest rate rise from i to i^* that starts in M years, continuous time:

$$\left[\vartheta\int_0^M e^{-ij}e^{-\vartheta j}dj+\vartheta\int_M^\infty e^{-iM-i^*(j-M)}e^{-\vartheta j}dj\right]\frac{B_t}{P_t^*}=\frac{s}{r}$$

$$\vartheta\left[\frac{e^{-(i+\vartheta)M}}{i^*+\vartheta}+\frac{1-e^{-(i+\vartheta)M}}{i+\vartheta}\right]\frac{B_t}{P_t^*}=\frac{s}{r}$$

$$\frac{P_t^*}{P_t}=\left(\frac{e^{-(i+\vartheta)M}}{i^*+\vartheta}+\frac{1-e^{-(i+\vartheta)M}}{i+\vartheta}\right)/\left(\frac{1}{i+\vartheta}\right)$$

$$\frac{P_t^*}{P}-1=e^{-(i+\vartheta)M}\left(\frac{i+\vartheta}{i^*+\vartheta}-1\right)\approx e^{-\vartheta M}\left(\frac{i+\vartheta}{i^*+\vartheta}-1\right)$$

An interest rate rise from i to i^* that only lasts M years, discrete time:

$$\left[\sum_{j=0}^{M-1}\frac{\theta^j}{(1+i^*)^j}+\sum_{j=M}^{\infty}\frac{\theta^M}{(1+i^*)^M}\frac{\theta^{(j-M)}}{(1+i)^{(j-M)}}\right]\frac{B_{t-1}}{P_t^*}=\frac{s}{1-\beta}$$

$$\left[\frac{1-\left(\frac{\theta}{1+i^*}\right)^M}{1-\frac{\theta}{1+i^*}}+\frac{\left(\frac{\theta}{1+i^*}\right)^M}{1-\frac{\theta}{1+i}}\right]\frac{B_{t-1}}{P_t^*}=\frac{s}{1-\beta}$$

Thus,

$$\frac{P_t^*}{P_t}-1=\left(\frac{1-\left(\frac{\theta}{1+i^*}\right)^M}{1-\frac{\theta}{1+i^*}}+\frac{\left(\frac{\theta}{1+i^*}\right)^M}{1-\frac{\theta}{1+i}}\right)/\left(\frac{1}{1-\frac{\theta}{1+i}}\right)-1$$

$$\frac{P_t^*}{P_t}-1=\left(1-\left(\frac{\theta}{1+i^*}\right)^M\right)\left(\frac{\frac{1}{1-\frac{\theta}{1+i^*}}}{\frac{1}{1-\frac{\theta}{1+i}}}-1\right)$$

$$\frac{P_t^*}{P_t}-1=\left(1-\left(\frac{\theta}{1+i^*}\right)^M\right)\left(\frac{1+i^*}{1+i_t}\frac{1+i-\theta}{1+i^*-\theta}-1\right)$$

$$\frac{P_t^*}{P_t}-1\approx\left(1-\theta^M\right)\left(\frac{1+i-\theta}{1+i^*-\theta}-1\right)$$

An interest rate rise from i to i^* that starts in M years, discrete time:

$$\left[\sum_{j=0}^{M-1}\frac{\theta^j}{(1+i)^j}+\sum_{j=M}^{\infty}\frac{\theta^M}{(1+i)^M}\frac{\theta^{(j-M)}}{(1+i^*)^{(j-M)}}\right]\frac{B_{t-1}}{P_t^*}=\frac{s}{1-\beta}$$

$$\left[\frac{1-\left(\frac{\theta}{1+i}\right)^M}{1-\frac{\theta}{1+i}}+\frac{\left(\frac{\theta}{1+i}\right)^M}{1-\frac{\theta}{1+i^*}}\right]\frac{B_{t-1}}{P_t^*}=\frac{s}{1-\beta}$$

Thus,

$$\frac{P_t^*}{P_t}-1=\left(\frac{1-\left(\frac{\theta}{1+i}\right)^M}{1-\frac{\theta}{1+i}}+\frac{\left(\frac{\theta}{1+i}\right)^M}{1-\frac{\theta}{1+i^*}}\right)/\left(\frac{1}{1-\frac{\theta}{1+i}}\right)-1$$

$$\frac{P_t^*}{P_t}-1=\left(\frac{\theta}{1+i}\right)^M\left(\frac{1}{1-\frac{\theta}{1+i^*}}-\frac{1}{1-\frac{\theta}{1+i}}\right)\left(\frac{1}{1-\frac{\theta}{1+i}}\right)$$

$$\frac{P_t^*}{P_t}-1=\left(\frac{\theta}{1+i}\right)^M\left(\frac{1+i^*}{1+i}\frac{1+i-\theta}{1+i^*-\theta}-1\right)$$

$$\frac{P_t^*}{P_t}-1\approx\theta^M\left(\frac{1+i-\theta}{1+i^*-\theta}-1\right)$$

9.2 Sticky-price model solution

Here I derive the explicit solutions (62)-(63), for inflation and output given the equilibrium path of interest rates. The simple model (58)-(59) is

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1})$$
$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t.$$

The model with money generalizes the IS equation only, to (73)

$$x_t = E_t x_{t+1} + (\sigma - \xi)\left(\frac{m}{c}\right) E_t\left[\left(i_{t+1} - i^m_{t+1}\right) - \left(i_t - i^m_t\right)\right] - \sigma\left(i_t - E_t \pi_{t+1}\right).$$

We can treat the two cases simultaneously by defining

$$z_t \equiv i_t - \left(\frac{\sigma - \xi}{\sigma}\right)\left(\frac{m}{c}\right) E_t\left[\left(i_{t+1} - i^m_{t+1}\right) - \left(i_t - i^m_t\right)\right]$$

and writing the IS equation as

$$x_t = E_t x_{t+1} - \sigma\left(z_t - E_t \pi_{t+1}\right).$$

One must be careful that lags of z_t are lags of expected interest rate changes, not lags of actual interest rate changes.

Expressing the model in lag operator notation,

$$E_t(1 - L^{-1})x_t = \sigma E_t L^{-1}\pi_t - \sigma z_t$$

$$E_t(1 - \beta L^{-1})\pi_t = \kappa x_t$$

Forward-differencing the second equation,

$$E_t(1 - L^{-1})(1 - \beta L^{-1})\pi_t = E_t(1 - L^{-1})\kappa x_t$$

Then substituting,

$$E_t(1 - L^{-1})\left(1 - \beta L^{-1}\right)\pi_t = \sigma\kappa E_t L^{-1}\pi_t - \sigma\kappa z_t$$

$$E_t\left[(1 - L^{-1})\left(1 - \beta L^{-1}\right) - \sigma\kappa L^{-1}\right]\pi_t = -\sigma\kappa z_t$$

$$E_t\left[1 - (1 + \beta + \sigma\kappa) L^{-1} + \beta L^{-2}\right]\pi_t = -\sigma\kappa z_t.$$

Factor the lag polynomial

$$E_t(1-\lambda_1 L^{-1})(1-\lambda_2 L^{-1})\pi_t = -\sigma\kappa z_t$$

where

$$\lambda_i = \frac{(1+\beta+\sigma\kappa) \pm \sqrt{(1+\beta+\sigma\kappa)^2 - 4\beta}}{2}.$$

Since $\lambda_1 > 1$ and $\lambda_2 < 1$, reexpress the result as

$$E_t\left[(1-\lambda_1^{-1}L)(1-\lambda_2 L^{-1})\lambda_1 L^{-1}\pi_t\right] = \sigma\kappa z_t$$

$$E_t\left[(1-\lambda_1^{-1}L)(1-\lambda_2 L^{-1})\pi_{t+1}\right] = \sigma\kappa\lambda_1^{-1} z_t$$

The bounded solutions are

$$\pi_{t+1} = E_{t+1}\frac{\lambda_1^{-1}}{(1-\lambda_1^{-1}L)(1-\lambda_2 L^{-1})}\sigma\kappa z_t + \frac{1}{(1-\lambda_1^{-1}L)}\delta_{t+1}$$

where δ_{t+1} is a sequence of unpredictable random variables, $E_t\delta_{t+1} = 0$. I follow the usual practice and I rule out solutions that explode in the forward direction.

Using a partial fractions decomposition to break up the right hand side,

$$\frac{\lambda_1^{-1}}{\left(1-\lambda_1^{-1}L\right)\left(1-\lambda_2 L^{-1}\right)} = \frac{1}{\lambda_1-\lambda_2}\left(1+\frac{\lambda_1^{-1}L}{1-\lambda_1^{-1}L}+\frac{\lambda_2 L^{-1}}{1-\lambda_2 L^{-1}}\right).$$

So,

$$\pi_{t+1} = \frac{1}{\lambda_1-\lambda_2}E_{t+1}\left(1+\frac{\lambda_1^{-1}L}{1-\lambda_1^{-1}L}+\frac{\lambda_2 L^{-1}}{1-\lambda_2 L^{-1}}\right)\sigma\kappa z_t + \frac{1}{(1-\lambda_1^{-1}L)}\delta_{t+1}$$

or in sum notation,

$$\pi_{t+1} = \sigma\kappa\frac{1}{\lambda_1-\lambda_2}\left(z_t + \sum_{j=1}^{\infty}\lambda_1^{-j}z_{t-j} + \sum_{j=1}^{\infty}\lambda_2^{j}E_{t+1}z_{t+j}\right) + \sum_{j=0}^{\infty}\lambda_1^{-j}\delta_{t+1-j}. \tag{88}$$

We can show directly that the long-run impulse-response function is 1:

$$\frac{1}{(1-\lambda_1^{-1})(1-\lambda_2)}\frac{\sigma\kappa}{\lambda_1} = -\frac{\sigma\kappa}{(1-\lambda_1)(1-\lambda_2)}$$

$$= -\frac{\sigma\kappa}{(1-(\lambda_1+\lambda_2)+\lambda_1\lambda_2)} = -\frac{\sigma\kappa}{(1-(1+\beta+\sigma\kappa)+\beta)} = 1.$$

Having found the path of π_t, we can find output by

$$\kappa x_t = \pi_t - \beta E_t \pi_{t+1}.$$

In lag operator notation, and shifting forward one period,

$$\kappa x_{t+1} = E_{t+1}\left[(1-\beta L^{-1})\pi_{t+1}\right]$$

$$\kappa x_{t+1} = \frac{\sigma\kappa}{\lambda_1-\lambda_2}E_{t+1}\left[(1-\beta L^{-1})\left(1+\frac{\lambda_1^{-1}L}{1-\lambda_1^{-1}L}+\frac{\lambda_2 L^{-1}}{1-\lambda_2 L^{-1}}\right)z_t\right]+E_{t+1}\frac{(1-\beta L^{-1})}{(1-\lambda_1^{-1}L)}\delta_{t+1}.$$

We can rewrite the polynomials to give

$$\kappa x_{t+1} = \frac{\sigma\kappa}{\lambda_1-\lambda_2}E_{t+1}\left[\frac{1-\beta\lambda_1^{-1}}{1-\lambda_1^{-1}L}+\frac{(1-\beta\lambda_2^{-1})\left(\lambda_2 L^{-1}\right)}{1-\lambda_2 L^{-1}}\right]z_t + E_{t+1}\left[\frac{1-\beta\lambda_1^{-1}}{1-\lambda_1^{-1}L}\right]\delta_{t+1}.$$

(In the second term, I use $E_t\left[\beta L^{-1}\delta_{t+1}\right]=0$) or, in sum notation,

$$\kappa x_{t+1} = \frac{\sigma\kappa}{\lambda_1-\lambda_2}\left[\left(1-\beta\lambda_1^{-1}\right)\sum_{j=0}^{\infty}\lambda_1^{-j}z_{t-j}+\left(1-\beta\lambda_2^{-1}\right)\sum_{j=1}^{\infty}\lambda_2^{j}E_{t+1}z_{t+j}\right]+$$

$$+\left(1-\beta\lambda_1^{-1}\right)\sum_{j=0}^{\infty}\lambda_1^{-j}\delta_{t+1-j}.$$

9.3 Impulse response function – explicit solution

The solution (88) is

$$\pi_{t+1} = \frac{\sigma\kappa}{\lambda_1-\lambda_2}\left(i_t+\sum_{j=1}^{\infty}\lambda_1^{-j}i_{t-j}+E_{t+1}\sum_{j=1}^{\infty}\lambda_2^{j}i_{t+j}\right)+\sum_{j=0}^{\infty}\lambda_1^{-j}\delta_{t+1-j}$$

$$\lambda_1 = \frac{(1+\beta+\sigma\kappa)+\sqrt{(1+\beta+\sigma\kappa)^2-4\beta}}{2}$$

$$\lambda_1 = \frac{(1+\beta+\sigma\kappa)-\sqrt{(1+\beta+\sigma\kappa)^2-4\beta}}{2}$$

While it is straightforward to calculate and simulate the solution for a given path of interest rates, it is useful also to have a formula for the response to a step function. We want to find the impulse-response function to $i_t = 0$, $t < 0$, and $i_t = i$, $t = 0, 1, 2, \ldots$ The interest rate rise is announced at time $-M$, so only $\delta_{-M} \neq 0$. That response is,

$$t < -(M+1): \; \pi_{t+1} = 0$$
$$-(M+1) \leq t \leq 0: \pi_{t+1} = \frac{\sigma\kappa}{\lambda_1 - \lambda_2}\left(\frac{\lambda_2^{-t}}{1-\lambda_2}\right) + \lambda_1^{-(t+1+M)}\delta_{-M}$$
$$0 < t: \pi_{t+1} = \frac{\sigma\kappa}{\lambda_1 - \lambda_2}\left(\frac{1}{1-\lambda_2} + \frac{\lambda_1^{-1}(1-\lambda_1^{-t})}{1-\lambda_1^{-1}}\right) + \lambda_1^{-(t+1+M)}\delta_{-M}$$

Proceeding in the same way, the solution for x is

$$\kappa x_{t+1} = \frac{\sigma\kappa}{\lambda_1 - \lambda_2}\left((1-\beta\lambda_1^{-1})\sum_{j=0}^{\infty}\lambda_1^{-j}i_{t-j} + (1-\beta\lambda_2^{-1})E_{t+1}\sum_{j=1}^{\infty}\lambda_2^{j}i_{t+j}\right) + (1-\beta\lambda_1^{-1})\sum_{j=0}^{\infty}\lambda_1^{-j}\delta_{t+1-j}$$

so the impulse-response function to $i_t = 0$, $t < 0$, and $i_t = i$, $t = 0, 1, 2, ..$announced at time $-M$, is,

$$t < -(M+1): \; x_{t+1} = 0$$
$$-(M+1) \leq t \leq -1: \kappa x_{t+1} = \frac{\sigma\kappa}{\lambda_1 - \lambda_2}(1-\beta\lambda_2^{-1})\frac{\lambda_2^{-t+1}}{1-\lambda_2} + (1-\beta\lambda_1^{-1})\lambda_1^{-(t+1+M)}\delta_{-M}$$
$$0 \leq t: \kappa x_{t+1} = \frac{\sigma\kappa}{\lambda_1 - \lambda_2}\left((1-\beta\lambda_1^{-1})\frac{1-\lambda_1^{-(t+1)}}{1-\lambda_1^{-1}} + (1-\beta\lambda_2^{-1})\frac{\lambda_2}{1-\lambda_2}\right) + \lambda_1^{-(t+1+M)}\delta_{-M}$$

The interest rate is then

$$r_t = i_t - E_t\pi_{t+1}.$$

For the impulse-response function, the expected and actual values are the same, except at $-M$, where though $\pi_{-M} \neq 0$, $E_{-M-1}\pi_{-M} = 0$. Hence,

$$t \leq -(M+1): \; r_t = 0 \tag{89}$$
$$-M \leq t < 0: r_t = -\frac{\sigma\kappa}{\lambda_1 - \lambda_2}\left(\frac{\lambda_2^{-t}}{1-\lambda_2}\right) - \lambda_1^{-(t+1+M)}\delta_{-M} \tag{90}$$
$$t = 0: r_t = i - \frac{\sigma\kappa}{\lambda_1 - \lambda_2}\left(\frac{\lambda_2^{-t}}{1-\lambda_2}\right) - \lambda_1^{-(t+1+M)}\delta_{-M} \tag{91}$$
$$0 < t: r_t = i - \frac{\sigma\kappa}{\lambda_1 - \lambda_2}\left(\frac{1}{1-\lambda_2} + \frac{\lambda_1^{-1}(1-\lambda_1^{-t})}{1-\lambda_1^{-1}}\right) - \lambda_1^{-(t+1+M)}\delta_{-M} \tag{92}$$

9.4 Three-equation model solution

I solve the three-equation model of Figure 15 by standard methods, incorporating the Taylor rule in to monetary policy rather than conditioning on the equilibrium interest rate and then constructing the underlying Taylor rule. Both methods give the same answer, but a conventional calculation is more transparent in this case, and it verifies that both approaches give the same answer.

While one can solve the model quickly via matrix techniques, here I use lag operator techniques to write the solution for inflation analytically.

The model is

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1})$$
$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t$$
$$i_t = \phi \pi_t + \hat{v}_t^i$$
$$v_t^i = \rho v_{t-1}^i + \varepsilon_t^i$$

Substituting the Taylor rule,

$$x_t = E_t x_{t+1} - \sigma(\phi \pi_t + v_t^i - E_t \pi_{t+1})$$
$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t$$

Expressing the model in lag operator notation,

$$E_t(1 - L^{-1}) x_t = \sigma E_t \left(L^{-1} - \phi\right) \pi_t - \sigma v_t^i$$

$$E_t(1 - \beta L^{-1}) \pi_t = \kappa x_t$$

Forward-differencing the second equation,

$$E_t(1 - L^{-1})(1 - \beta L^{-1}) \pi_t = E_t(1 - L^{-1}) \kappa x_t$$

Then substituting into the first equation,

$$E_t(1 - L^{-1}) \left(1 - \beta L^{-1}\right) \pi_t = \sigma \kappa E_t \left(L^{-1} - \phi\right) \pi_t - \sigma \kappa v_t^i$$

$$E_t\left[1-\frac{1+\beta+\sigma\kappa}{1+\sigma\kappa\phi}L^{-1}+\frac{\beta}{1+\sigma\kappa\phi}L^{-2}\right]\pi_t=-\frac{\sigma\kappa}{1+\sigma\kappa\phi}v_t^i.$$

Factor the lag polynomial

$$E_t(1-\lambda_1L^{-1})(1-\lambda_2L^{-1})\pi_t=-\frac{\sigma\kappa}{1+\sigma\kappa\phi}v_t^i$$

where

$$\lambda=\frac{1+\beta+\sigma\kappa\pm\sqrt{(1+\beta+\kappa\sigma)^2-4\beta\left(1+\phi\sigma\kappa\right)}}{2\left(1+\sigma\kappa\phi\right)}$$

These lag operator roots are the inverse of the eigenvalues of the usual transition matrix. The system is stable and solved backward for $\lambda>1$; it is unstable and solved forward for $\lambda<1$.

The standard three-equation model uses $\phi>1$ so both roots are unstable, $\lambda_1<1$ and $\lambda_2<1$. Then, we can write

$$E_t(1-\lambda_1L^{-1})(1-\lambda_2L^{-1})\pi_t=-\frac{\sigma\kappa}{1+\sigma\kappa\phi}v_t^i$$

$$\pi_t=-E_t\frac{1}{(1-\lambda_1L^{-1})(1-\lambda_2L^{-1})}\frac{\sigma\kappa}{1+\sigma\kappa\phi}v_t^i$$

$$\pi_t=E_t\frac{1}{\lambda_1-\lambda_2}\left(\frac{-\lambda_1}{1-\lambda_1L^{-1}}+\frac{\lambda_2}{1-\lambda_2L^{-1}}\right)\frac{\sigma\kappa}{1+\sigma\kappa\phi}v_t^i$$

$$\pi_t=\frac{\sigma\kappa}{1+\sigma\kappa\phi}\frac{1}{\lambda_1-\lambda_2}E_t\left(-\lambda_1\sum_{j=0}^{\infty}\lambda_1^jv_{t+j}^i+\lambda_2\sum_{j=0}^{\infty}\lambda_2^jv_{t+j}^i\right)$$

Using the AR(1) form of the disturbance v^i,

$$\pi_t=\frac{\sigma\kappa}{1+\sigma\kappa\phi}\frac{1}{\lambda_1-\lambda_2}\left(-\lambda_1\sum_{j=0}^{\infty}\lambda_1^j\rho^j+\lambda_2\sum_{j=0}^{\infty}\lambda_2^j\rho^j\right)\hat{\imath}_t$$

$$\pi_t=\frac{\sigma\kappa}{1+\sigma\kappa\phi}\frac{1}{\lambda_1-\lambda_2}\left(-\frac{\lambda_1}{1-\lambda_1\rho}+\frac{\lambda_2}{1-\lambda_2\rho}\right)v_t^i$$

$$\pi_t=\frac{\sigma\kappa}{1+\sigma\kappa\phi}\frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_2\left(1-\lambda_1\rho\right)-\lambda_1\left(1-\lambda_2\rho\right)}{\left(1-\lambda_1\rho\right)\left(1-\lambda_2\rho\right)}\right)v_t^i$$

$$\pi_t=-\frac{\sigma\kappa}{1+\sigma\kappa\phi}\left(\frac{1}{\left(1-\lambda_1\rho\right)\left(1-\lambda_2\rho\right)}\right)v_t^i$$

Thus, to produce Figure 15, I simply simulate the AR(1) impulse-response, for $\{v_t^i\}$, calculate π_t by the last equation, and calculate $i_t=\phi\pi_t+v_t^i$.

9.5 Impulse-response with long-term debt and price stickiness

I develop the exact nonlinear formulas for the value of surpluses and a linear approximation. The linear approximation turns out to be quite accurate in this application.

An interest rate rise from time $t = 0$ onwards is announced at time $t = -M$. I calculate for each value of the inflation shock δ_{-M} the percent change in a constant surplus corresponding to that shock. Writing the surplus as $se^{\Delta s}$, the value of nominal debt before the shock satisfies

$$\sum_{j=0}^{\infty}\left(\prod_{k=0}^{j-1}\frac{1}{1+f^{(k)}}\right)\frac{B^{(j)}}{P_{-M}} = \sum_{t=-M}^{\infty}\beta^{t-M}s. = \frac{1}{1-\beta}s.$$

where $\left\{B^{(j)}\right\}$ is the observed maturity structure of the debt, and the observed forward rates are $f^{(j)}$, ($f_t^{(j)}$ is the forward rate at time t for loans from $t+j$ to $t+j+1$; $f_t^{(0)} = i_t$ is the one-period interest rate). After the shock, nominal interest rates increase by i, the price level jumps from P_{-M} to P^*_{-M}, with

$$e^{\pi^*_{-M}} = e^{\pi_{-M}+\delta_{-M}} = \frac{P^*_{-M}}{P_{-M}}.$$

Here, π_{-M} denotes the solution with $\delta = 0$, so actual inflation after the shock is announced is $\pi^*_{-M} = \pi_{-M} + \delta_{-M}$. The basic solution for inflation (88) includes a jump in inflation when the shock is announced, and I have defined δ as additional unexpected changes in inflation. Surpluses rise to $se^{\Delta s}$, giving

$$\sum_{j=0}^{\infty}\left(\prod_{k=0}^{M-1}\frac{1}{1+f^{(k)}}\right)\left(\prod_{k=M}^{j-1}\frac{1}{1+f^{(k)}+i}\right)\frac{B^{(j)}}{P^*_{-M}} = \sum_{t=-M}^{\infty}\beta^{(t-M)}\frac{u'(C_t)}{u'(C_{-M})}se^{\Delta s} \tag{93}$$

To easily calculate the multiperiod discount factor on the right hand side, I use

$$\frac{u'(C_t)}{u'(C_\tau)} = \frac{e^{-\gamma(c+x_t)}}{e^{-\gamma(c+x_\tau)}} = e^{-\frac{1}{\sigma}(x_t-x_\tau)}$$

Dividing pre and post shock values of (93), s cancels and

$$e^{\pi_{-M}+\delta_{-M}} = \frac{\sum_{j=0}^{\infty}\left(\prod_{k=0}^{M-1}\frac{1}{1+f^{(k)}}\right)\left(\prod_{k=M}^{j-1}\frac{1}{1+f^{(k)}+i}\right)B^{(j)}}{\sum_{j=0}^{\infty}\left(\prod_{k=0}^{j-1}\frac{1}{1+f^{(k)}}\right)B^{(j)}}\frac{\sum_{t=-M}^{\infty}\beta^{(t-M)}}{\sum_{t=-M}^{\infty}\beta^{(t-M)}e^{-\frac{1}{\sigma}(x_t-x_{-M})}}e^{-\Delta s}.$$

Conversely, then, we can find the surplus required to support a given time -M shock δ_{-M} – whether that surplus comes from active or from passive fiscal policy – by solving for Δs,

$$e^{\Delta s}=\frac{\sum_{j=0}^{\infty}\left(\prod_{k=0}^{M-1}\frac{1}{1+f^{(k)}}\right)\left(\prod_{k=M}^{j-1}\frac{1}{1+f^{(k)}+i}\right)B^{(j)}}{\sum_{j=0}^{\infty}\left(\prod_{k=0}^{j-1}\frac{1}{1+f^{(k)}}\right)B^{(j)}}\frac{\sum_{t=-M}^{\infty}\beta^{(t-M)}}{\sum_{t=-M}^{\infty}\beta^{(t-M)}e^{-\frac{1}{\sigma}(x_t-x_{-M})}}e^{-(\pi_{-M}+\delta_{-M})}. \tag{94}$$

For each choice of δ_{-M}, then, I find the solution for inflation and interest rates by (90)-(92); I compute the product of real rates in the bottom right term of (94), and I compute the required percentage change in surplus Δs. To find the fiscal-theory / long-term debt solution, I search for the δ_{-M} that produces $\Delta s=0$. It is important to treat the numerator and denominator of the last term of (94) equally. If one truncates the denominator, truncate the numerator at the same point.

9.6 Linearized valuation equation

To linearly approximate (94), write

$$e^{\Delta s}\approx V\frac{\sum_{t=-M}^{\infty}\beta^{(t-M)}}{\sum_{t=-M}^{\infty}\beta^{(t-M)}e^{-\frac{1}{\sigma}(x_t-x_{-M})}}e^{-(\pi_{-M}+\delta_{-M})}. \tag{95}$$

$$1+\Delta s\approx(1+v)\frac{\sum_{t=-M}^{\infty}\beta^{(t-M)}}{\sum_{t=-M}^{\infty}\beta^{(t-M)}\left(1-\frac{1}{\sigma}(x_t-x_{-M})\right)}(1-(\pi_{-M}+\delta_{-M})). \tag{96}$$

$$1+\Delta s\approx(1+v)\frac{1}{1-\frac{\sum_{t=-M}^{\infty}\beta^{(t-M)}\frac{1}{\sigma}(x_t-x_{-M})}{\sum_{t=-M}^{\infty}\beta^{(t-M)}}}(1-(\pi_{-M}+\delta_{-M})). \tag{97}$$

$$\Delta s\approx v+\frac{\sum_{t=-M}^{\infty}\beta^{(t-M)}\frac{1}{\sigma}(x_t-x_{-M})}{\sum_{t=-M}^{\infty}\beta^{(t-M)}}-(\pi_{-M}+\delta_{-M}). \tag{98}$$

$$\Delta s\approx v+(1-\beta)\sum_{t=-M}^{\infty}\beta^{(t-M)}\frac{1}{\sigma}(x_t-x_{-M})-(\pi_{-M}+\delta_{-M}). \tag{99}$$

In numerical experimentation, it turns out that the exact and linearized approach produce almost exactly the same answer to the first few decimals. So, the nonlinearity of long-term present values is not an issue for this magnitude – a few percent at most – of interest rate variation.

For one-period debt, v is unchanged so we have

$$\Delta s \approx -\Delta E_t(\pi_t) + \frac{1-\beta}{\sigma}\sum_{j=0}^{\infty}\beta^j \Delta E_t(x_{t+j} - x_t) \tag{100}$$

where $\Delta E_t \equiv E_t - E_{t-1}$ and t is the date of the announcement of a new policy.

The first term of (100) captures the fact that unexpected inflation devalues outstanding government debt. In the second term, $(x_{t+j} - x_t)/\sigma$ is the real interest rate between time t and time $t+j$. So this term captures the fact that if real rates rise, the government must pay more interest on the debt.

9.7 The Model with Money

This section derives the model with money (73). The utility function is

$$\max E \int_{t=0}^{\infty} e^{-\delta t} u(c_t, M_t/P_t) dt.$$

The present-value budget constraint is

$$\frac{B_0 + M_0}{P_0} = \int_{t=0}^{\infty} e^{-\int_{s=0}^{t} r_s ds}\left[c_t - y_t + s_t + (i_t - i_t^m)\frac{M_t}{P_t}\right] dt$$

where

$$r_t = i_t - \frac{dP_t}{P_t}$$

and s denotes real net taxes paid, and thus the real government primary surplus. This budget constraint is the present value form of

$$d(B_t + M_t) = i_t B_t + i_t^m M_t + P_t(y_t - c_t - s_t).$$

Introducing a multiplier λ on the present value budget constraint, we have

$$\frac{\partial}{\partial c_t} : e^{-\delta t} u_c(t) = \lambda e^{-\int_{s=0}^{t} r_s ds},$$

where (t) means $(c_t, M_t/P_t)$. Differentiating with respect to time,

$$-\delta e^{-\delta t} u_c(t) + e^{-\delta t} u_{cc}(t)\frac{dc_t}{dt} + e^{-\delta t} u_{cm}(t)\frac{dm_t}{dt} = -\lambda r_t\, e^{-\int_{s=0}^{t} r_s ds}$$

where $m_t \equiv M_t/P_t$. Dividing by $e^{-\delta t}u_c(t)$, we obtain the intertemporal first order condition:

$$-\frac{c_t u_{cc}(t)}{u_c(t)}\frac{dc_t}{c_t} - \frac{m_t u_{cm}(t)}{u_c(t)}\frac{dm_t}{m_t} = (r_t - \delta)\,dt. \tag{101}$$

The first-order condition with respect to M is

$$\begin{aligned}\frac{\partial}{\partial M_t} : e^{-\delta t} u_m(t)\frac{1}{P_t} &= \lambda e^{-\int_{s=0}^{t} r_s ds}\left(i_t - i_t^m\right)\frac{1}{P_t}\\ e^{-\delta t} u_m(t) &= e^{-\delta t} u_c(t)\left(i_t - i_t^m\right)\\ \frac{u_m(t)}{u_c(t)} &= i_t - i_t^m. \end{aligned} \tag{102}$$

The last equation is the usual money demand curve.

Thus, an equilibrium $c_t = y_t$ satisfies

$$\frac{-c_t u_{cc}(t)}{u_c(t)}\frac{dc_t}{c_t} - \frac{m_t u_{cm}(t)}{u_c(t)}\frac{dm_t}{m_t} = -\delta dt + \left(i_t - \frac{dP_t}{P_t}\right)dt \tag{103}$$

$$\frac{u_m(t)}{u_c(t)} = i_t - i_t^m \tag{104}$$

$$\frac{B_0 + M_0}{P_0} = \int_{t=0}^{\infty} e^{-\int_{s=0}^{t} r_s ds}\left[s_t + \left(i_t - i_t^m\right)\frac{M_t}{P_t}\right]dt \tag{105}$$

The last equation combines the consumer's budget constraint and equilibrium $c = y$. I call it the government debt valuation formula.

9.7.1 CES functional form

I use a standard money in the utility function specification with a CES functional form,

$$u(c_t, m_t) = \frac{1}{1-\gamma}\left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}}.$$

I use the notation $m = M/P$, with capital letters for nominal and lowercase letters for real quantities.

This CES functional form nests three important special cases. Perfect substitutes is the case $\theta = 0$:

$$u(c_t, m_t) = \frac{1}{1-\gamma}\left[c_t + \alpha m_t\right]^{1-\gamma}.$$

The Cobb-Douglas case is $\theta \longrightarrow 1$:

$$u(c_t, m_t) \longrightarrow \frac{1}{1-\gamma}\left[c_t^{\frac{1}{1+\alpha}} m_t^{\frac{\alpha}{1+\alpha}}\right]^{1-\gamma}. \tag{106}$$

The monetarist limit is $\theta \to \infty$:

$$u(c_t, m_t) \to \frac{1}{1-\gamma}\left[\min\left(c_t, \alpha m_t\right)\right]^{1-\gamma}.$$

I call it the monetarist limit because money demand is then $M_t/P_t = c_t/\alpha$, i.e. $\alpha = 1/V$ is constant, and the interest elasticity is zero. The separable case is $\theta = \gamma$:

$$u(c_t, m_t) = \frac{1}{1-\gamma}\left[c_t^{1-\gamma} + \alpha m_t^{1-\gamma}\right].$$

In the separable case, u_c is independent of m, so money has no effect on the intertemporal substitution relation, and hence on inflation and output dynamics in a new-Keynesian model under an interest rate target. Terms in $(\theta - \gamma)$ or $(\sigma - \xi)$ with $\sigma = 1/\gamma$ and $\xi = 1/\theta$ will characterize deviations from the separable case, how much the marginal utility of consumption is affected by money.

With this functional form, the derivatives are

$$u_c = \left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]^{\frac{\theta-\gamma}{1-\theta}} c_t^{-\theta}$$

$$u_m = \left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]^{\frac{\theta-\gamma}{1-\theta}} \alpha m_t^{-\theta}.$$

Equilibrium condition (104) becomes

$$\frac{u_m(t)}{u_c(t)} = \alpha\left(\frac{m_t}{c_t}\right)^{-\theta} = i_t - i_t^m. \tag{107}$$

The second derivative with respect to consumption is

$$\frac{u_{cc}}{u_c} = (\theta-\gamma)\frac{1}{\left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]} c_t^{-\theta} - \theta c_t^{-1}$$

$$-\frac{c u_{cc}}{u_c} = -\frac{(\theta-\gamma)\, c_t^{1-\theta} - \theta\left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]}{\left[c_t^{1-\theta} + \alpha m_t^{1-\theta}\right]}$$

$$-\frac{cu_{cc}}{u_c} = \frac{\gamma c_t^{1-\theta} + \theta\alpha m_t^{1-\theta}}{c_t^{1-\theta} + \alpha m_t^{1-\theta}}$$

$$-\frac{cu_{cc}}{u_c} = \gamma \frac{\left[1 + \frac{\theta}{\gamma}\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}\right]}{\left[1 + \alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}\right]}.$$

The cross derivative is

$$\frac{mu_{cm}}{u_c} = (\theta - \gamma)\frac{\alpha m_t^{1-\theta}}{c_t^{1-\theta} + \alpha m_t^{1-\theta}}$$
$$= (\theta - \gamma)\frac{\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}{1 + \alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}.$$

or, using (107)

$$\frac{mu_{cm}}{u_c} = (\theta - \gamma)\frac{\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1 + \left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}.$$

9.7.2 Money demand

Money demand (107) can be written

$$\frac{m_t}{c_t} = \left(\frac{1}{\alpha}\right)^{-\xi}(i_t - i_t^m)^{-\xi}. \tag{108}$$

where $\xi = 1/\theta$ becomes the interest elasticity of money demand, in log form, and α governs the overall level of money demand.

The steady state obeys

$$\frac{m}{c} = \left(\frac{1}{\alpha}\right)^{-\xi}(i - i^m)^{-\xi}. \tag{109}$$

so we can write money demand (108) in terms of steady state real money as

$$\frac{m_t}{c_t} = \left(\frac{m}{c}\right)\left(\frac{i_t - i_t^m}{i - i^m}\right)^{-\xi}, \tag{110}$$

avoiding the parameter α. (Throughout, numbers without time subscripts denote steady state values.)

The product $\frac{m}{c}(i - i^m)$, the interest cost of holding money, appears in many subsequent expressions. It is

$$\frac{m}{c}(i-i^m) = \left(\frac{1}{\alpha}\right)^{-\xi}(i-i^m)^{1-\xi}.$$

With $\xi < 1$, as interest rates go to zero this interest cost goes to zero as well.

9.7.3 Intertemporal Substitution

The first order condition for the intertemporal allocation of consumption (103) is

$$-\frac{c_t u_{cc}(t)}{u_c(t)}\frac{dc_t}{c_t} - \frac{m_t u_{cm}(t)}{u_c(t)}\frac{dm_t}{m_t} = -\delta dt + (i_t - \pi_t)\, dt$$

where $\pi_t = dP_t/P_t$ is inflation. This equation shows us how, with nonseparable utility, monetary policy can distort the allocation of consumption over time, in a way not captured by the usual interest rate effect. That is the central goal here. In the case of complements, $u_{cm} > 0$ (more money raises the marginal utility of consumption), larger money growth makes it easier to consume in the future relative to the present, and acts like a higher interest rate, inducing higher consumption growth.

Substituting in the CES derivatives,

$$\gamma\frac{1+\frac{\theta}{\gamma}\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}{1+\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}\frac{dc_t}{c_t} - (\theta-\gamma)\frac{\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}{1+\alpha\left(\frac{m_t}{c_t}\right)^{1-\theta}}\frac{dm_t}{m_t} = -\delta dt + (i_t - \pi_t)\, dt$$

and using (107) to eliminate α

$$\gamma\frac{1+\frac{\theta}{\gamma}\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1+\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}\frac{dc_t}{c_t} - (\theta-\gamma)\frac{\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1+\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}\frac{dm_t}{m_t} = -\delta dt + (i_t - \pi_t)\, dt \qquad (111)$$

We can make this expression prettier as

$$\gamma\frac{dc_t}{c_t} + (\theta-\gamma)\frac{\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1+\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}\left(\frac{dc_t}{c_t} - \frac{dm_t}{m_t}\right) = -\delta dt + (i_t - \pi_t)\, dt$$

Rexpressing in terms of the intertemporal substitution elasticity $\sigma = 1/\gamma$ and interest elasticity of money demand $\xi = 1/\theta$, and multiplying by σ,

$$\frac{dc_t}{c_t} + \left(\frac{\sigma - \xi}{\xi}\right) \frac{\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1 + \left(\frac{m_t}{c_t}\right)(i_t - i_t^m)} \left(\frac{dc_t}{c_t} - \frac{dm_t}{m_t}\right) = -\delta\sigma dt + \sigma\left(i_t - \pi_t\right) dt. \tag{112}$$

We want to substitute interest rates for money. To that end, differentiate the money demand curve

$$\frac{m_t}{c_t} = \left(\frac{m}{c}\right)\left(\frac{i_t - i_t^m}{i - i^m}\right)^{-\xi}$$

$$\frac{m_t}{c_t}\left(\frac{dm_t}{m_t} - \frac{dc_t}{c_t}\right) = -\xi\left(\frac{m}{c}\right)\left(\frac{i_t - i_t^m}{i - i^m}\right)^{-\xi} \frac{d\left(i_t - i_t^m\right)}{i_t - i_t^m}$$

$$\left(\frac{dc_t}{c_t} - \frac{dm_t}{m_t}\right) = \xi \frac{\frac{m}{c}}{\frac{m_t}{c_t}}\left(\frac{i_t - i_t^m}{i - i^m}\right)^{-\xi} \frac{d\left(i_t - i_t^m\right)}{i_t - i_t^m}$$

Substituting,

$$\frac{dc_t}{c_t} + \left(\frac{\sigma - \xi}{\xi}\right) \frac{\left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}{1 + \left(\frac{m_t}{c_t}\right)(i_t - i_t^m)} \left(\xi \frac{\frac{m}{c}}{\frac{m_t}{c_t}}\left(\frac{i_t - i_t^m}{i - i^m}\right)^{-\xi} \frac{d\left(i_t - i_t^m\right)}{i_t - i_t^m}\right) = -\delta\sigma dt + \sigma\left(i_t - \pi_t\right) dt.$$

$$\frac{dc_t}{c_t} + (\sigma - \xi)\frac{m}{c}\frac{1}{1 + \left(\frac{m_t}{c_t}\right)(i_t - i_t^m)}\left(\frac{i_t - i_t^m}{i - i^m}\right)^{-\xi} d\left(i_t - i_t^m\right) = -\delta\sigma dt + \sigma\left(i_t - \pi_t\right) dt.$$

With $x_t = \log c_t$, $dx_t = dc_t/c_t m$, approximating around a steady state, and approximating that the interest cost of holding money is small, $\left(\frac{m}{c}\right)(i - i^m) << 1$, we obtain the intertemporal substitution condition modified by interest costs,

$$\frac{dx_t}{dt} + (\sigma - \xi)\frac{m}{c}\frac{d\left(i_t - i_t^m\right)}{dt} = \sigma\left(i_t - \pi_t\right). \tag{113}$$

In discrete time,

$$E_t x_{t+1} - x_t + (\sigma - \xi)\left(\frac{m}{c}\right)\left[E_t\left(i_{t+1} - i_{t+1}^m\right) - \left(i_t - i_t^m\right)\right] = \sigma\left(i_t - E_t \pi_{t+1}\right).$$

For models with monetary control, one wants an IS curve expressed in terms of the monetary aggregate. From (112), with the same approximations and $\tilde{m} = \log(m)$,

$$\frac{dx_t}{dt} + \left(\frac{\sigma - \xi}{\xi}\right)\left(\frac{m}{c}\right)(i - i^m)\left(\frac{dx_t}{dt} - \frac{d\tilde{m}_t}{dt}\right) = \sigma\left(i_t - \pi_t\right) dt. \tag{114}$$

In discrete time,

$$(E_t x_{t+1} - x_t) + \left(\frac{\sigma - \xi}{\xi}\right)\left(\frac{m}{c}\right)(i - i^m)\left[(E_t x_{t+1} - x_t) - E_t\left(\tilde{m}_{t+1} - \tilde{m}_t\right)\right] = \sigma\left(i_t - \pi_t\right). \quad (115)$$