Appendix: A Framework for Debt-Maturity Management

Contents

1 Introduction 1

2 Maturity management with liquidity costs 8
   2.1 Model setup ......................................................... 8
   2.2 Solution: the debt issuance rule ................................ 11
   2.3 Asymptotic behavior ............................................. 13
   2.4 The cases without liquidity costs and vanishing liquidity costs .................. 15
   2.5 Calibration .......................................................... 16
   2.6 Maturity management with unexpected shocks ......................... 19

3 Risk 24
   3.1 The model with risk ............................................... 24
   3.2 Solution: risk-adjusted valuations ................................ 25
   3.3 The risky steady state ............................................. 28

4 Default 30
   4.1 The option to default ............................................. 31
   4.2 Default-adjusted valuations ..................................... 32
   4.3 The impact of default on maturity choice ......................... 35

5 Extensions 38
   5.1 Alternative specifications for the liquidity costs .................. 38
   5.2 Finite issuances ..................................................... 39
   5.3 Consols ............................................................... 39

6 Conclusions 41

A Equivalence between PDE and integral formulations 2

B Micro model of liquidity costs 4
   B.1 Environment ......................................................... 4
   B.2 Valuations .......................................................... 4
   B.3 Solution .............................................................. 5
A Equivalence between PDE and integral formulations

Valuations and prices are given by continuous-time net present value formulae. Their PDE representation is the analogue of the recursive representation in discrete time and the integral formulation is the equivalent of the sequence summations. The solutions to each PDE can be recovered easily via the method of characteristics or as an immediate application of the Feynman-Kac formula. All of the PDEs in this paper have an exact solution contained in Table A.
<table>
<thead>
<tr>
<th>Price (PF)</th>
<th>PDE</th>
<th>$r^*(t)\psi(t,t) = \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau}; \psi(0,t) = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integral</td>
<td>$e^{-\int_t^T r_s(u)du} + \delta \int_t^T e^{-\int_s^T r_s(u)du}ds$</td>
<td></td>
</tr>
<tr>
<td>Valuation (PF)</td>
<td>PDE</td>
<td>$r(t)v(t,t) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}; v(0,t) = 1$</td>
</tr>
<tr>
<td>Integral</td>
<td>$e^{-\int_t^T r_s(u)du} + \delta \int_t^T e^{-\int_s^T r_s(u)du}ds$</td>
<td></td>
</tr>
<tr>
<td>Price (risk)</td>
<td>PDE</td>
<td>$\dot{p}^*(t)\phi(t,t) = \delta + \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial \tau} + \phi X^t[\psi(t,\tau,\xi) - \hat{\psi}(\tau,t)]; \hat{\psi}(0,t) = 1$</td>
</tr>
<tr>
<td>Integral</td>
<td>$e^{-\int_t^T \phi^*(u)du} + \int_t^T \left(\delta + \phi\psi(t,\tau,\xi) - \phi X^t[\psi(t,\tau,\xi) - \hat{\psi}(\tau,t)]; \hat{\psi}(0,t) = 1\right)$</td>
<td></td>
</tr>
<tr>
<td>Valuation (risk)</td>
<td>PDE</td>
<td>$\dot{v}^*(t)\phi(t,t) = \delta + \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial \tau} + \phi X^t[\psi(t,\tau,\xi) - \hat{\psi}(\tau,t)]; \hat{\psi}(0,t) = 1$</td>
</tr>
<tr>
<td>Integral</td>
<td>$e^{-\int_t^T \phi^*(u)du} + \int_t^T \left(\delta + \phi\psi(t,\tau,\xi) - \phi X^t[\psi(t,\tau,\xi) - \hat{\psi}(\tau,t)]; \hat{\psi}(0,t) = 1\right)$</td>
<td></td>
</tr>
<tr>
<td>Price (default)</td>
<td>PDE</td>
<td>$\dot{r}^*(t)\phi(t,t) = \delta + \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial \tau} + \phi X^t[\psi(t,\tau,\xi) - \hat{\psi}(\tau,t)]; \hat{\psi}(0,t) = 1$</td>
</tr>
<tr>
<td>Integral</td>
<td>$e^{-\int_t^T \phi^*(u)du} + \int_t^T \left(\delta + \phi\psi(t,\tau,\xi) - \phi X^t[\psi(t,\tau,\xi) - \hat{\psi}(\tau,t)]; \hat{\psi}(0,t) = 1\right)$</td>
<td></td>
</tr>
<tr>
<td>Valuation (default)</td>
<td>PDE</td>
<td>$\dot{r}^*(t)\phi(t,t) = \delta + \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial \tau} + \phi X^t[\psi(t,\tau,\xi) - \hat{\psi}(\tau,t)]; \hat{\psi}(0,t) = 1$</td>
</tr>
<tr>
<td>Integral</td>
<td>$e^{-\int_t^T \phi^*(u)du} + \int_t^T \left(\delta + \phi\psi(t,\tau,\xi) - \phi X^t[\psi(t,\tau,\xi) - \hat{\psi}(\tau,t)]; \hat{\psi}(0,t) = 1\right)$</td>
<td></td>
</tr>
<tr>
<td>Debt Profile</td>
<td>PDE</td>
<td>$\frac{\partial f}{\partial t} = \tau(t,t) + \frac{\partial f}{\partial \tau}$</td>
</tr>
<tr>
<td>Integral</td>
<td>$f(t,t) = \int_t^{\min{T,T\tau+\tau}} \lambda(s,t+s)ds + |T &gt; t + \tau| \cdot f(0,t+\tau)$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Equivalence between PDE and integral formulations.
B  Micro model of liquidity costs

B.1 Environment

Here we describe the microeconomic model of the liquidity costs in more detail. This is a wholesale retail model of the secondary market of sovereign bonds. Without loss of generality, we focus on the issuance of \(i(\tau, t)\)-bonds at time \(t\) with maturity \(\tau\). We define \(s\) as the amount of time passed since the auction. The outstanding amount of bonds in the hands of an atomistic banker, after a period of time \(s\) has passed after the issuance of the bond is:

\[
I \left( s; i(\tau, t) \right) = \max\left( i(\tau, t) - \mu y_{ss} \cdot s, 0 \right).
\]

This implies that the bond inventory is exhausted by time:

\[
s = \frac{i(\tau, t)}{\mu y_{ss}}.
\]

We consider that individual orders arrive randomly according to a Poisson distribution. The intensity at which bonds are sold per unit of time is given by:

\[
\gamma^{(t, \tau)}(s) = \frac{\mu y_{ss}}{I \left( s; i(\tau, t) \right)} = \frac{1}{s - s}\text{ for } s \in [0, \min\{\tau, \bar{s}\}).
\]

This intensity \(\gamma^{(t, \tau)}(s)\) is defined only between \([0, \min\{\tau, \bar{s}\})\), because after the bond matures or after the stock is exhausted, there is no further selling.

B.2 Valuations

**Investor’s valuation.** At time \(t + s\), after a period of time \(s\) has passed since the auction, the time to maturity is \(\tau' = \tau - s\). The valuation of the bond by investors is defined as:

\[
\psi^{(t, \tau)}(\tau', s) \equiv \psi(\tau - s, t + s).
\]

They are risk neutral and discount future payoffs at the international market rate. Hence, the price equation satisfies the PDE (2.4):

\[
r^s(t + s)\psi^{(t, \tau)}(\tau', s) = \delta - \frac{\partial\psi^{(t, \tau)}}{\partial\tau'} + \frac{\partial\psi^{(t, \tau)}}{\partial t},
\]

with the terminal condition of \(\psi^{(t, \tau)}(0, s) = 1\).

**Banker’s valuation.** Now consider the valuation of the cash flows of the bond from the perspective of the banker \(q^{(t, \tau)}(\tau', s)\). Bankers are risk neutral but have a higher cost of capital. At each moment \(t + s\) bankers meet investors and sell at a price \(\psi^{(t, \tau)}(\tau', s)\). The valuation of the bankers, \(q^{(t, \tau)}(\tau', s)\) satisfies:

\[
(r^s(t + s) + \eta)q^{(t, \tau)}(\tau', s) = \delta - \frac{\partial q^{(t, \tau)}}{\partial\tau'} + \frac{\partial q^{(t, \tau)}}{\partial t} + \gamma^{(t, \tau)}(s) \left( \psi^{(t, \tau)}(\tau', s) - q^{(t, \tau)}(\tau', s) \right). \tag{B.1}
\]

This expression takes this form because the banker extracts surplus \(\left( \psi^{(t, \tau)}(\tau', s) - q^{(t, \tau)}(\tau', s) \right)\) when he is matched to an investor. Before a match, bankers earn the flow utility, but upon a match, their value jumps to \(\psi^{(t, \tau)} - q^{(t, \tau)}\). This jump arrives with endogenous intensity \(\gamma^{(t, \tau)}(s)\). The complication with this PDE is its terminal condition.
Thus, the approximate liquidity cost function is $\lambda$. If $\bar{q}$, we solve the PDE for $\psi$. The solution to $q(t, \tau, t, t)$ is:

$$\lambda(t, \tau, t) = \psi(t, \tau, t) - q(t, \tau, t),$$

is the object we are trying to find.

**B.3 Solution**

We now provide a first order linear approximation for the price at the auction, $q(t, \tau, t)$, for small issuances. The result is given by the following proposition:

**Proposition 4.** A first-order Taylor expansion around $t = 0$ yields a linear auction price:

$$q(t, \tau, t) \approx \psi(t, \tau, t) - \frac{1}{2} \frac{\eta}{s} \psi(t, \tau, t)\mu(s, t).$$

(B.2)

Thus, the approximate liquidity cost function is $\lambda(t, \tau, t) \approx \frac{1}{2} \bar{\lambda} \psi(t, \tau, t)\mu(s, t)$ where the price impact is given by $\bar{\lambda} = \frac{\eta}{s} \mu(s, t)$.

**Proof.** Step 1. Exact solutions. The solution to $q(t, \tau, t)$ falls into one of two cases. Case 1. If $\bar{s} \leq \tau$, then:

$$q(t, \tau, t) = \int_{0}^{\tau} e^{-\int_{0}^{\tau} (r(t) + \eta) \, du} \left( \delta(\bar{s} - v) + \psi(t - v, t + v) \right) \, dv.$$

(B.3)

Case 2. If $\bar{s} > \tau$, then:

$$q(t, \tau, t) = \int_{0}^{\tau} e^{-\int_{0}^{\tau} (r(t) + \eta) \, du} \left( \delta(\bar{s} - v) + \psi(t - v, t + v) \right) \, dv + e^{-\int_{0}^{\tau} (r(t) + \eta) \, du} \frac{\bar{s} - \tau}{\bar{s}}.$$

(B.4)

We solve the PDE for $q$ depending on the corresponding terminal conditions, $q(t, \tau, s) = \psi(t, \tau, s)$ and $q(t, \tau, 0) = 1$.

Case 1. Consider the first case. The general solution to the PDE equation for $q(t, \tau, s)$ is,

$$\int_{0}^{\bar{s} - s} e^{-\int_{0}^{\tau} (r(t) + \eta + \gamma(u)) \, du} \left( \delta + \gamma(s + v) \psi(t - v, t + v) \right) \, dv + e^{-\int_{0}^{s} (r(t) + \eta + \gamma(u)) \, du} \psi(t' - (\bar{s} - s), t + (\bar{s} - s)).$$

(B.5)

This can be checked by taking partial derivatives with respect to time and maturity and applying Leibniz’s rule.\(^{26}\) Consider the exponentials that appear in both terms of equation (B.5). These can be decomposed into

\(^{26}\)Notice that we have directly replaced the value $\psi(t, \tau, s) = \psi(t - s, t + s)$.\(\)
When we evaluate this expression at \( s \),

\[
e^{-\int_0^\tau r(t+u)du}e^{-\int_0^\tau \gamma(u)du}.
\]

Then, by definition of \( \gamma \) we have:

\[
e^{-\int_0^\tau \gamma(u)du} = e^{-\int_0^\tau \frac{1}{s-u}du} = \frac{(s-v)}{s}.
\]

Thus, using (B.6) in (B.5) we can re-express it as:

\[
q^{(t,\tau)}(\tau',s) = \int_0^{\tilde{\tau}-\tau} e^{-\int_0^\tau (r(t+u)+\gamma(u))du} \left( \frac{(s-v)}{s} \delta + \frac{\psi(\tau - v, t + v)}{s} \right) dv
\]

When we evaluate this expression at \( s = 0, \tau' = \tau \), and we replace \( \gamma(v) = \frac{1}{s-v} \), we arrive at:

\[
q(t,\tau,t) \equiv q^{(t,\tau)}(\tau,0)
= \int_0^\tau e^{-\int_0^\tau (r(t+u)+\gamma(u))du} \left( \frac{(s-v)}{s} \delta + \frac{\psi(\tau - v, t + v)}{s} \right) dv.
\]

**Case 2.** The proof in the second case runs parallel to Case 1 above. The general solution to the PDE in this case is:

\[
q^{(t,\tau)}(\tau',s) = \int_0^{\tau'} e^{-\int_0^\tau (r(t+u)+\gamma(u))du} \left( \frac{(s-v)}{s} \delta + \frac{\psi(\tau - v, t + v)}{s} \right) dv
\]

When we evaluate this expression at \( s = 0, \tau' = \tau \):

\[
q(t,\tau,t) = \int_0^\tau e^{-\int_0^\tau (r(t+u)+\gamma(u))du} \left( \frac{(s-v)}{s} \delta + \frac{\psi(\tau - v, t + v)}{s} \right) dv
\]

\[
+ e^{-\int_0^\tau (r(t+u)+\gamma(u))du} \frac{s - \tau}{s}.
\]

**Step 2. Limit Behavior of \( q(t,\tau,t) \).** Price with zero issuances. Consider the limit \( \lim_{\tau \to 0} q(t,\tau,t) \) for any \( \tau \). For both Case 1 and Case 2, equations (B.3) and (B.4),\(^{27}\) it holds that:

\[
\lim_{\tau \to 0} q(t,\tau,t) = \lim_{\tilde{\tau} \to 0} \int_0^\tilde{\tau} e^{-\int_0^\tau (r(t+u)+\gamma(u))du} \left( \delta(\tilde{\tau} - s) + \psi(\tau - s, t + s) \right) d\tilde{\tau}.
\]

Now, both the numerator and the denominator converge to zero as we take the limits. Hence, by L’Hôpital’s rule, the limit of the price is the limit of the ratio of derivatives. The derivative of the numerator is obtained via Leibniz’s rule and thus,

\[
\lim_{\tau \to 0} q(t,\tau,t) = \lim_{s \to \tilde{\tau}} \left[ \frac{e^{-\int_0^\tau (r(t+u)+\gamma(u))du} (\delta(\tilde{\tau} - s) + \psi(\tau - s, t + s))}{1} \right]_{s=\tilde{\tau}}
= \lim_{s \to \tilde{\tau}} e^{-\int_0^\tau (r(t+u)+\gamma(u))du} \psi(\tau - s, t + s)
\]

\[
= \psi(\tau, t).
\]

**Step 3. Linear approximation of \( q(t,\tau,t) \).** The first order approximation of the function \( q(t,\tau,t) \), the price at the

\(^{27}\) For every \( \tau < \tilde{\tau} \), i.e. in Case 2, it will be analogous since we are taking the limit when \( \tilde{\tau} \) converges to zero.
auction, around $t = 0$ is given by:

$$q(i, \tau, t) \simeq q(i, \tau, t) \big|_{t=0} + \left. \frac{\partial q(i, \tau, t)}{\partial t} \right|_{t=0} i(\tau, t).$$

We computed the first term in step 2. It is given by $\psi(\tau, t)$. Thus, our objective will be to obtain $\left. \frac{\partial q(i, \tau, t)}{\partial t} \right|_{t=0}$. Observe that by definition of $\bar{s}$, it holds that:

$$\frac{\partial q(i, \tau, t)}{\partial t} = \frac{\partial \bar{s}}{\partial t} \frac{\partial q(i, \tau, t)}{\partial \bar{s}} = \frac{1}{\mu y_{ss}} \frac{\partial q(i, \tau, t)}{\partial \bar{s}},$$

where we have applied the fact that $\bar{s} = \frac{\langle \tau, t \rangle}{\mu y_{ss}}$. For further reference, note that

$$\left. \frac{\partial q(i, \tau, t)}{\partial t} \right|_{t=0} = \lim_{\bar{s} \to 0} \frac{\partial q(i, \tau, t)}{\partial \bar{s}} \frac{1}{\mu y_{ss}}. \quad (B.7)$$

Step 3.1. Derivative $\frac{\partial q(i, \tau, t)}{\partial \bar{s}}$. Consider the price function corresponding to Case 1. The derivative of the price function with respect to $\bar{s}$ is given by:

$$\frac{\partial q(i, \tau, t)}{\partial \bar{s}} = \frac{\partial}{\partial \bar{s}} \left( \int_0^\bar{s} e^{-\int_0^u (r^*(t+u)+\eta)du} \left( \delta(\bar{s} - s) + \psi(t - \bar{s}, t + \bar{s}) \right) ds \right)$$

$$= \frac{\partial}{\partial \bar{s}} \left( \frac{\int_0^\bar{s} e^{-\int_0^u (r^*(t+u)+\eta)du} \psi(t - \bar{s}, t + \bar{s}) + \int_0^\bar{s} \delta e^{-\int_0^u (r^*(t+u)+\eta)du} ds}{\bar{s}} \right)$$

$$= \frac{e^{-\int_0^\bar{s} (r^*(t+u)+\eta)du} \psi(t - \bar{s}, t + \bar{s}) + \int_0^\bar{s} \delta e^{-\int_0^u (r^*(t+u)+\eta)du} ds - q(i, \tau, t)}{\bar{s}}. \quad (B.8)$$

Note that in the last line we used the definition of $q(i, \tau, t)$ as given for Case 1.

Step 3.2. Re-writing the limit of $\frac{\partial q(i, \tau, t)}{\partial \bar{s}}$. To obtain $\left. \frac{\partial q(i, \tau, t)}{\partial \bar{s}} \right|_{t=0}$ we compute $\lim_{\bar{s} \to 0} \frac{\partial q(i, \tau, t)}{\partial \bar{s}}$ using equation (B.8). In equation (B.8) both the numerator and denominator converge to zero as $\bar{s} \to 0$. Thus, we employ L’Hôpital’s rule to obtain the derivative of interest. The derivative of the denominator is 1. Thus, the limit of (B.8) is now given by:

$$\lim_{\bar{s} \to 0} \frac{\partial q(i, \tau, t)}{\partial \bar{s}} = \lim_{\bar{s} \to 0} \frac{\partial}{\partial \bar{s}} \left[ e^{-\int_0^\bar{s} (r^*(t+u)+\eta)du} \psi(t - \bar{s}, t + \bar{s}) + \int_0^\bar{s} \delta e^{-\int_0^u (r^*(t+u)+\eta)du} ds - q(i, \tau, t) \right]. \quad (B.9)$$

28The limits of the three terms in the numerator of equation (B.8) are respectively:

$$\lim_{\bar{s} \to 0} \int_0^\bar{s} e^{-\int_0^u (r(t+u)+\eta)du} ds = 0,$$

$$\lim_{\bar{s} \to 0} e^{-\int_0^\bar{s} (r(t+u)+\eta)du} \psi(t - \bar{s}, t + \bar{s}) = \psi(t, t),$$

$$\lim_{\bar{s} \to 0} q(i, \tau, t) = \psi(\tau, t).$$
Step 3.3. Consider the first two terms of \((B.9)\). Applying Leibniz’s rule:

\[
\lim_{\tilde{s} \to 0} \left\{ \left( -\frac{\partial}{\partial \tau} \psi(\tau - \tilde{s}, t + \tilde{s}) + \frac{\partial}{\partial t} \psi(\tau - \tilde{s}, t + \tilde{s}) - (r^*(t + \tilde{s}) + \eta)\psi(\tau - \tilde{s}, t + \tilde{s}) \right) e^{-\int_0^\tilde{s} (r^*(t + u) + \eta)du} + \delta e^{-\int_0^\tilde{s} (r^*(t + u) + \eta)du} \right\}.
\]

The previous limit is given by:

\[
-\frac{\partial}{\partial \tau} \psi(\tau, t) + \frac{\partial}{\partial t} \psi(\tau, t) - (r^*(t) + \eta)\psi(\tau, t) + \delta.
\]

Using the valuation of the international investors, we can rewrite the previous equation as:

\[
-\frac{\partial}{\partial \tau} \psi(\tau, t) + \frac{\partial}{\partial t} \psi(\tau, t) - (r^*(t) + \eta)\psi(\tau, t) + \delta = r^*(t)\psi(\tau, t) - (r^*(t) + \eta)\psi(\tau, t)
\]

\[
= -\eta\psi(\tau, t).
\]

This the first two terms of the limit of \(\frac{\partial q(i, \tau, t)}{\partial s}\) are equal to \(-\eta\psi(\tau, t)\). Computing the limit of \(\frac{\partial q(i, \tau, t)}{\partial s}\): last term. The last term of \((B.9)\) is given by

\[
-\lim_{s \to 0} \frac{\partial q(i, \tau, t)}{\partial s} = -\lim_{s \to 0} \frac{\partial q(i, \tau, t)}{\partial t} \frac{\partial t}{\partial s} = -\frac{\partial q(i, \tau, t)}{\partial t} \bigg|_{s=0} \mu y_{ss},
\]

where we used \((B.7)\). Thus, from \((B.10)\) and \((B.11)\), the derivative \((B.8)\) is given by:

\[
\lim_{s \to 0} \frac{\partial q(i, \tau, t)}{\partial s} = -\frac{\partial q(i, \tau, t)}{\partial t} \bigg|_{s=0} \mu y_{ss} - \eta\psi(\tau, t).
\]

Plugging \((B.12)\) in \((B.7)\) we obtain that:

\[
\frac{\partial q(i, \tau, t)}{\partial t} \bigg|_{i=0} = \left( -\mu y_{ss} \frac{\partial q(i, \tau, t)}{\partial i} \bigg|_{i=0} - \eta\psi(\tau, t) \right) - \frac{1}{\mu y_{ss}}.
\]

Rearranging terms, we conclude that:

\[
\frac{\partial q(i, \tau, t)}{\partial t} \bigg|_{i=0} = -\frac{\eta\psi(\tau, t)}{2\mu y_{ss}}.
\]

Step 4. Taylor expansion. A first-order Taylor expansion around zero emissions yields:

\[
q(i, \tau, t) = q(i, \tau, t) \bigg|_{i=0} + \frac{\partial q(i, \tau, t)}{\partial i} \bigg|_{i=0} i(\tau, t),
\]

\[
= \psi(\tau, t) - \frac{\eta\psi(\tau, t)}{2\mu y_{ss}} i(\tau, t),
\]

where we used \((B.13)\). We can define price impact as \(\lambda = \frac{\eta}{\mu y_{ss}}\). This concludes the proof. \(\square\)
C Proofs

C.1 Proof of Proposition 1

Proof. First we construct a Lagrangian on the space of functions $g$ such that are Lebesgue integrable, $\left\lVert e^{-\rho t/2} g (\tau, t) \right\rVert^2 < \infty$. The Lagrangian, after replacing $c(t)$ from the budget constraint, is:

$$
\mathcal{L} [u, f] = \int_0^\infty e^{-\rho t} U \left( y(t) - f(0, t) + \int_0^T [g(\tau, t, t) \ell(\tau, t) - \delta f(\tau, t)] d\tau \right) dt
$$

$$
+ \int_0^\infty \int_0^T e^{-\rho t} j(\tau, t) \left( -\frac{\partial f}{\partial t} + \ell(\tau, t) + \frac{\partial f}{\partial \tau} \right) d\tau dt,
$$

where $j(\tau, t)$ is the Lagrange multiplier associated to the law of motion of debt.

We consider a perturbation $h(\tau, t), e^{-\rho t} h \in L^2 ([0, T] \times [0, \infty))$, around the optimal solution. Since the initial distribution $f_0$ is given, any feasible perturbation must satisfy $h(\tau, 0) = 0$. In addition, we know that $f(T, t) = 0$ because $f(T^+, t) = 0$ (by construction) and issuances are infinitesimal. Thus, any admissible variation must also feature $h(T, t) = 0$. At an optimal solution $f$, the Lagrangian must satisfy $\mathcal{L} [u, f] \geq \mathcal{L} [u, f + ah]$ for any perturbation $h(\tau, t)$.

Taking the derivative with respect to $a$ — i.e., computing the Gâteaux derivative, for any suitable $h(\tau, t)$ we obtain:

$$
\frac{d}{da} \mathcal{L} [u, f + ah] \bigg|_{a=0} = \int_0^\infty e^{-\rho t} U' (c(t)) \left( -h(0, t) - \int_0^T \delta h(\tau, t) d\tau \right) dt
$$

$$
- \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j(\tau, t) d\tau dt
$$

$$
+ \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial \tau} j(\tau, t) d\tau dt.
$$

We employ integration by parts to show that:

$$
\int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j(\tau, t) d\tau dt = \int_0^T \int_0^\infty e^{-\rho t} \frac{\partial h}{\partial t} j(\tau, t) d\tau dt
$$

$$
= \int_0^\infty \left( \lim_{s \to \infty} e^{-\rho s} h(\tau, s) j(\tau, s) \right) d\tau
$$

$$
- \int_0^\infty \int_0^T e^{-\rho t} \left( \frac{\partial j(\tau, t)}{\partial t} - \rho j(\tau, t) \right) h(\tau, t) d\tau dt,
$$

and

$$
\int_0^\infty e^{-\rho t} \int_0^T \frac{\partial h}{\partial \tau} j(\tau, t) d\tau dt = \int_0^\infty e^{-\rho t} \left[ h(T, t) j(T, t) - h(0, t) j(0, t) - \int_0^T h(\tau, t) \frac{\partial j}{\partial \tau} d\tau \right] dt.
$$
Replacing these calculations in the Lagrangian, and equating it to zero, yields:

\[
0 = \int_0^\infty e^{-\rho t} U'(c(t)) \left[ -h(0, t) - \int_0^T \delta h(\tau, t) d\tau \right] dt \\
+ \int_0^\infty \int_0^T e^{-\rho t} \left( -\rho j - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t} \right) h(\tau, t) d\tau dt \\
+ \int_0^\infty e^{-\rho t} (h(T, t) j(T, t) - h(0, t) j(0, t)) dt \\
- \int_0^\infty \lim_{s \to -\infty} e^{-\rho s} h(\tau, s) j(\tau, s) d\tau + h(\tau, 0) j(\tau, 0).
\]

We rearrange terms to obtain:

\[
0 = -\int_0^\infty e^{-\rho t} \left[ U'(c(t)) - j(0, t) \right] h(0, t) dt \\
+ \int_0^\infty \int_0^T e^{-\rho t} \left( -\rho j - U'(c) \delta - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t} \right) h(\tau, t) d\tau dt \\
- \int_0^\infty e^{-\rho t} (h(T, t) j(T, t)) dt \\
- \int_0^\infty \lim_{s \to -\infty} e^{-\rho s} h(\tau, s) j(\tau, s) d\tau + h(\tau, 0) j(\tau, 0).
\]  

(C.1)

Since \( h(T, t) = h(\tau, 0) = 0 \) is a condition for any admissible variation, then, both the third line in equation (C.1) and the second term in the fourth line are equal to zero. Furthermore, because (C.1) needs to hold for any feasible variation \( h(\tau, t) \), all the terms that multiply \( h(\tau, t) \) should equal zero. The latter, yields a system of necessary conditions for the Lagrange multipliers:

\[
\rho j(\tau, t) = -\delta U'(c(t)) + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \text{ if } \tau \in (0, T],
\]

\[
j(0, t) = -U'(c(t)), \text{ if } \tau = 0,
\]

\[
\lim_{t \to -\infty} e^{-\rho t} j(\tau, t) = 0, \text{ if } \tau \in (0, T].
\]

(C.2)

Next, we perturb the control. We proceed in a similar fashion:

\[
\frac{d}{d\alpha} L[i + \alpha h, f] \bigg|_{\alpha=0} = \int_0^\infty e^{-\rho t} U'(c(t)) \left[ \int_0^T \left( \frac{\partial q}{\partial t} i(\tau, t) + q(\tau, t, i) \right) h(\tau, t) d\tau \right] dt \\
+ \int_0^\infty \int_0^T e^{-\rho t} h(\tau, t) j(\tau, t) d\tau dt.
\]

Collecting terms and setting the Lagrangian to zero, we obtain:

\[
\int_0^\infty \int_0^T e^{-\rho t} \left[ j(\tau, t) + U'(c(t)) \left( \frac{\partial q}{\partial t} i(\tau, t) + q(\tau, t, i) \right) \right] h(\tau, t) d\tau dt = 0.
\]

Thus, setting the term in parenthesis to zero, amounts to setting:

\[
U'(c(t)) \left( \frac{\partial q}{\partial t} i(\tau, t) + q(\tau, t, i) \right) = -j(\tau, t).
\]

(C.3)
Next, we define the Lagrange multiplier in terms of goods:

\[ v(\tau, t) = -j(\tau, t) / U'(c(t)). \tag{C.4} \]

Taking the derivative of \( v(\tau, t) \) with respect to \( t \) and \( \tau \) we can express the necessary conditions, (C.2) in terms of \( v \). In particular, we transform the PDE in (C.2) into the summary equations in the Proposition. That is:

\[
\left( \rho - \frac{U''(c(t)) c(t) \dot{c}(t)}{U'(c(t))} \right) v(\tau, t) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0, T],
\]

\[
v(0, t) = 1, \text{ if } \tau = 0,
\]

\[
\lim_{t \to \infty} e^{-\rho t} v(\tau, t) = 0, \text{ if } \tau \in (0, T];
\]

and the first-order condition, (C.3), is now given by:

\[
\frac{\partial v}{\partial t} + q(\tau, t, \tau) v(\tau, t) = 0,
\]

as we intended to show. \( \square \)

### C.2 Duality

Given a path of resources \( y(t) \), the primal problem, the one solved in section 2, is given by:

\[
V[f(\cdot, 0)] = \max_{\{i(\tau), c(\tau)\}_{\tau \in [0, T]}} \int_0^\infty e^{-\rho(t-s)} u(c(s)) ds \text{ s.t.}
\]

\[
c(t) = y(t) - f(0, t) + \int_0^T [q(\tau, t, \tau) i(\tau, t) - \delta f(\tau, t)] d\tau
\]

\[
\frac{\partial f}{\partial t} = i(\tau, t) + \frac{\partial f}{\partial \tau}; f(\tau, 0) = f_0(\tau).
\]

Here we show that this problem has a dual formulation. This dual formulation, minimizes the resources needed to sustain a given path of consumption \( c(t) \):

\[
D[f(\cdot, 0)] = \min_{\{i(\tau), t(\tau)\}_{\tau \in [0, T]}} \int_0^\infty e^{-\rho(t-s)} y(s) ds \text{ s.t.}
\]

\[
c(t) = y(t) - f(0, t) + \int_0^T [q(\tau, t, \tau) i(\tau, t) - \delta f(\tau, t)] d\tau
\]

\[
\frac{\partial f}{\partial t} = i(\tau, t) + \frac{\partial f}{\partial \tau}; f(\tau, 0) = f_0(\tau)
\]

\[
r(t) = \rho - \frac{U''(c(t)) c(t) \dot{c}(t)}{U'(c(t)) c(t)}
\]

**Proposition 5.** Consider the solution \( \{c^*(t), i^*(\tau, t), f^*(\tau, t)\}_{t \geq 0, \tau \in (0, T]} \) to the Primal Problem given \( f_0 \). Then, given the path of consumption \( c^*(t) \), \( \{y^*(t), i^*(\tau, t), f^*(\tau, t)\}_{t \in [0, T], \tau \in (0, T]} \) solves the Dual Problem where:

\[
y^*(t) = c^*(t) + f^*(0, t) + \int_0^T [q(\tau, t, i^*) i^*(\tau, t) - \delta f^*(\tau, t)] d\tau.
\]

**Proof.** Step 1. We start following the steps of Proposition 1. We construct the Lagrangian for the Dual Problem in
Replacing these calculations in the Lagrangian, and equating it to zero, yields:

\[
\mathcal{L}[i,f] = \int_0^\infty e^{-\int_0^t \mathcal{L}(t) ds} \left( c(t) + f(0,t) - \int_0^T [q(\tau,t) \delta f(\tau,t)] d\tau \right) dt
\]

\[
+ \int_0^\infty \int_0^T e^{-\int_0^t \mathcal{L}(t) ds} v(\tau,t) \left( -\frac{\partial f}{\partial t} + \alpha(\tau,t) + \frac{\partial f}{\partial \tau} \right) d\tau dt,
\]

where \(v(\tau,t)\) is the Lagrange multiplier associated to the law of motion of debt. We again consider a perturbation \(h(\tau,t), e^{-\rho t} h \in L^2 ([0,T] \times [0,\infty))\), around the optimal solution. Recall that because \(f_0\) is given, and \(f(T,t) = 0\), any feasible perturbation needs to meet: \(h(\tau,0) = 0\) and \(h(\tau,t) = 0\). At an optimal solution \(f\), it must be the case that \(\mathcal{L}[i,f] \geq \mathcal{L}[i,f + \alpha h]\) for any feasible perturbation \(h(\tau,t)\). This implies that

\[
\frac{\partial}{\partial \alpha} \mathcal{L}[i,f + \alpha h] \bigg|_{\alpha = 0} = \int_0^\infty e^{-\int_0^t \mathcal{L}(t) ds} \left[ h(0,t) + \int_0^T \delta h(\tau,t) d\tau \right] dt
\]

\[
- \int_0^\infty \int_0^T e^{-\int_0^t \mathcal{L}(t) ds} \frac{\partial h}{\partial t} v(\tau,t) d\tau dt
\]

\[
+ \int_0^\infty \int_0^T e^{-\int_0^t \mathcal{L}(t) ds} \frac{\partial h}{\partial \tau} v(\tau,t) d\tau dt.
\]

We again employ integration by parts to show that:

\[
\int_0^\infty \int_0^T e^{-\int_0^t \mathcal{L}(t) ds} \frac{\partial h}{\partial t} v(\tau,t) d\tau dt = \int_0^T \int_0^\infty e^{-\int_0^t \mathcal{L}(t) ds} \frac{\partial h}{\partial t} dtd\tau
\]

\[
= \int_0^T \left( \lim_{s \to \infty} e^{-\int_0^t \mathcal{L}(t) ds} h(\tau,s) v(\tau,s) \right) - h(\tau,0) v(\tau,0) \right) d\tau
\]

\[
- \int_0^\infty \int_0^T e^{-\int_0^t \mathcal{L}(t) ds} \left( \frac{\partial v(\tau,t)}{\partial t} - r(t) v(\tau,t) \right) h(\tau,t) dtd\tau
\]

\[
= \int_0^T \left( \lim_{s \to \infty} e^{-\int_0^t \mathcal{L}(t) ds} h(\tau,s) v(\tau,s) - h(\tau,0) \right) v(\tau,0) \right) d\tau
\]

\[
- \int_0^\infty e^{-\int_0^t \mathcal{L}(t) ds} \int_0^T \left( \frac{\partial v(\tau,t)}{\partial t} - r(t) v(\tau,t) \right) h(\tau,t) d\tau dt,
\]

and

\[
\int_0^\infty e^{-\int_0^t \mathcal{L}(t) ds} \int_0^T \frac{\partial h}{\partial \tau} v(\tau,t) d\tau dt = \int_0^\infty e^{-\int_0^t \mathcal{L}(t) ds} \left[ h(T,t) v(T,t) - h(0,t) v(0,t) - \int_0^T h(\tau,t) \frac{\partial v(t)}{\partial \tau} d\tau \right] dt.
\]

Replacing these calculations in the Lagrangian, and equating it to zero, yields:
Again, the previous equation needs to hold for any feasible variation \( h(t, t) \), all the terms that multiply \( h(t, t) \) should be equal to zero. The latter, yields a system of necessary conditions for the Lagrange multipliers, and substituting for the value of \( r \):

\[
0 = \int_0^\infty e^{-\int_0^t r(s) ds} \left[ h(0, t) + \int_0^T \delta h(\tau, t) d\tau \right] dt + \int_0^\infty \int_0^T e^{-\int_0^t r(s) ds} \left( -r(t) v - \frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial t} \right) h(\tau, t) d\tau dt + \int_0^\infty e^{-\int_0^t r(s) ds} \left( h(T, t) v(T, t) - h(0, t) v(0, t) \right) dt - \int_0^\infty \lim_{\tau \to \infty} e^{-\int_0^\tau r(u) du} h(\tau, s) v(\tau, s) d\tau.
\]

By proceeding in a similar fashion with the control we arrive to:

\[
\left( \frac{\partial q}{\partial t} + q(t, t, t) \right) = -v(t, t).
\]

Note that system of equation (C.5) to (C.6) plus the budget constraint, the law of motion of debt, and initial debt \( f_0 \), are precisely the conditions that characterize the solution of the primal problem.

\[\square\]

C.3 Asymptotic behavior

Here we formally prove the limit conditions that we discussed after Proposition 1. In particular, we provide a complete asymptotic characterization. The following Proposition provides a summary.

**Proposition 6.** Assume that \( \rho > r_{ss}^* \), there exists a steady state if and only if \( \hat{\lambda} > \lambda_o \) for some \( \lambda_o \). If instead, \( \hat{\lambda} \leq \lambda_o \), there is no steady state but consumption converges asymptotically to zero. In particular, the asymptotic behavior is:

**Case 1 (High Liquidity Costs).** For liquidity costs above the threshold value \( \hat{\lambda} > \lambda_o \), variables converge to a steady state characterized by the following system:

\[
\begin{align*}
\dot{c}_{ss} &= 0, \\
\dot{r}_{ss} &= 0, \\
\lambda_{ss}(\tau) &= \frac{\psi_{ss}(\tau) - v_{ss}(\tau)}{\hat{\lambda} \psi_{ss}(\tau)}, \\
v_{ss}(\tau) &= \frac{\delta}{\rho} \left( 1 - e^{-\rho \tau} \right) + e^{-\rho \tau}, \\
f_{ss}(\tau) &= \int_\tau^T \lambda_{ss}(s) ds, \\
c_{ss} &= y_{ss} - f_{ss}(0) + \int_0^T \left[ \psi_{ss}(\tau) \lambda_{ss}(\tau) - \frac{\hat{\lambda} \psi_{ss}(\tau)}{2} \dot{\lambda}_{ss}(\tau)^2 - \delta f_{ss}(\tau) \right].
\end{align*}
\]
Case 2 (Low Liquidity Costs). For liquidity costs below the threshold value $0 < \bar{\lambda} \leq \bar{\lambda}_o$, variables converge asymptotically to:

$$
\lim_{s \to \infty} \frac{c(s)}{c(t)} = e^{-\frac{(r - r_0(\lambda))(s-t)}{\bar{\lambda}}} \\
v_{\infty}(\tau, r_\infty(\bar{\lambda})) = \frac{\delta}{r_{\infty}(\bar{\lambda})} (1 - e^{-r_0(\lambda)\tau}) + e^{-r_0(\lambda)\tau} \\
t_{\infty}(\tau, r_\infty(\bar{\lambda})) = \frac{\psi_{ss}(\tau) - v_{\infty}(\tau, r_\infty(\bar{\lambda}))}{\lambda \psi_{ss}(\tau)} \\
f_{\infty}(\tau, r_\infty(\bar{\lambda})) = \int_{\tau}^{T} t_{\infty}(s, r_\infty(\bar{\lambda})) ds
$$

where $r_\infty(\bar{\lambda})$ satisfies $r_{ss}^\lambda \leq r_\infty(\bar{\lambda}) < \rho$ and solves:

$$
c_\infty = 0 = y_{ss} - f_{\infty}(0, r_\infty(\bar{\lambda})) + \int_{0}^{T} \left[ t_{\infty}(\tau, r_\infty(\bar{\lambda})) \psi(\tau) - \frac{\bar{\lambda} \psi_{ss}(\tau)}{2} t_{\infty}(\tau, r_\infty(\bar{\lambda}))^2 - \delta f_{\infty}(\tau, r_\infty(\bar{\lambda})) \right] d\tau.
$$

Threshold. The threshold $\bar{\lambda}_o$ solves $|c_{ss}|_{\lambda=\bar{\lambda}_o} = 0$ in (C.11) and $\lim_{\bar{\lambda} \to \bar{\lambda}_o} r_\infty(\bar{\lambda}) = \rho$.

Proof. Step 1. First observe that as $\bar{\lambda} \to \infty$, the optimal issuance policy (2.12) approaches $i(\tau, t) = 0$. Thus, for that limit, $c_{ss} = y > 0$ and $f_{ss}(\tau) = 0$.

Step 2. Next, consider the system in Case 1 of Proposition 6 as a guess of a solution. Note that equations (C.8) to (C.11) meet the necessary conditions of Proposition 1 as long as $r(t) = \rho$. This because: $t_{ss}(\tau)$ meets the first order condition with respect to the control; $v_{ss}(\tau)$ solves the PDE for valuations; given $t_{ss}(\tau)$ and $v_{ss}(\tau)$ the stock of debt solves the KFE, thus, is given by $\int_{\tau}^{T} t_{ss}(s) ds$; and consumption is pinned down by the budget constraint. In addition, by construction, consumption determined in (C.11) does not depend on time; i.e. $c(t) = 0$ and this implies that

$$
r_{ss} \equiv r(t) = \rho.
$$

Thus, the only thing we need to check is that there exists some $\bar{\lambda}$ finite such that consumption is positive.

Step 3. The system in equations (C.8) to (C.11) is continuous in $\bar{\lambda}$. Therefore, because $c_{ss} = y > 0$ for $\bar{\lambda} \to \infty$, there exists a value of $\bar{\lambda}$ such that the implied consumption by equations (C.8) to (C.11) is positive.

Step 4. We now prove that there is an interval where this solution holds. In particular, we will show that $c_{ss}$ decreases as $\bar{\lambda}$ increases. Observe that, steady state internal valuations $v_{ss}(\tau)$ in (C.9) and bond prices $\psi(\tau)$ are independent of $\bar{\lambda}$. Steady-state debt issuance’s $t_{ss}(\tau)$ in (C.8) are a monotonously decreasing function of $\bar{\lambda}$, because

$$
\frac{\partial t_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}} t_{ss}(\tau) < 0,
$$

and therefore the total amount of debt at each maturity $f_{ss}(\tau)$ in (C.10) is also decreasing with $\bar{\lambda}$, because

$$
\frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}} f_{ss}(\tau) < 0.
$$
If we take derivatives with respect to $\bar{\lambda}$ in the budget constraint (C.11) we obtain:

$$\frac{dc_{ss}}{\partial \lambda} = -\frac{\partial f_{ss}(0)}{\partial \lambda} + \int_{0}^{T} \left[ \psi_{ss}(\tau) \frac{\partial t_{ss}(\tau)}{\partial \lambda} - \frac{\psi_{ss}(\tau)}{2} t_{ss}(\tau)^{2} - \bar{\lambda} \psi_{ss}(\tau) t_{ss}(\tau) \frac{\partial t_{ss}(\tau)}{\partial \lambda} - \bar{\lambda} \frac{\partial f_{ss}(\tau)}{\partial \lambda} \right] d\tau$$

$$= \frac{1}{\bar{\lambda}} f_{ss}(0) - \frac{1}{\bar{\lambda}} \int_{0}^{T} \left[ \psi_{ss}(\tau) t_{ss}(\tau) + \frac{\bar{\lambda}}{2} \psi_{ss}(\tau) t_{ss}(\tau)^{2} - \bar{\lambda} \psi_{ss}(\tau) t_{ss}(\tau)^{2} - \frac{\partial f_{ss}(\tau)}{\partial \lambda} \right] d\tau$$

$$= -\frac{1}{\bar{\lambda}} c_{ss} < 0.$$

Observe that $t_{ss}(\tau)$ can be made arbitrarily small by increasing $\bar{\lambda}$. Thus, there exists a value of $\bar{\lambda} \geq 0$ such that $c_{ss} = 0$ in the system above. We denote this value by $\bar{\lambda}_o$.

Step 5. For $\bar{\lambda} \leq \bar{\lambda}_o$, if a steady state existed, it would imply $c_{ss} < 0$, outside of the range of admissible values. Therefore, there is no steady state in this case. Assume that the economy grows asymptotically at rate $g_{\infty}(\bar{\lambda}) \equiv \lim_{t \to \infty} \frac{dc}{dt}$. If $g_{\infty}(\bar{\lambda}) > 0$ then consumption would grow to infinity, which violates the budget constraint. Thus, if there exists an asymptotic the growth rate, it is negative: $g_{\infty}(\bar{\lambda}) < 0$. If we define $r_{\infty}(\bar{\lambda})$ as

$$r_{\infty}(\bar{\lambda}) \equiv (\rho + \sigma g(\bar{\lambda})) < \rho,$$

the growth rate of the economy can be expressed as

$$g_{\infty}(\bar{\lambda}) = -\left(\frac{\rho - r_{\infty}(\bar{\lambda})}{\sigma}\right).$$

When this is the case, the asymptotic valuation is

$$v_{\infty}(\tau, r_{\infty}(\bar{\lambda})) = \frac{\delta (1 - e^{-r_{\infty}(\bar{\lambda})\tau})}{r_{\infty}(\bar{\lambda})} + e^{-r_{\infty}(\bar{\lambda})\tau}.$$ 

To obtain the discount factor bounds, observe that if $v_{\infty}(\tau, r_{\infty}(\bar{\lambda})) \leq \psi_{ss}(\tau)$ the optimal issuance is non-negative. Otherwise issuances would be negative at all maturities and the country would be an asymptotic net asset holder. This cannot be an optimal solution as this implies that consumption can be increased just by reducing the amount of foreign assets. Therefore, $r_{\infty}(\bar{\lambda}) \geq r^*$. Finally, by definition $r_{\infty}(\bar{\lambda}) < \rho$. □

C.4 No liquidity costs: $\bar{\lambda} = 0$

Proposition 7. (Optimal Policy with Liquid Debt) Assume that $\lambda(\tau, t, s) = 0$. If a solution exists, then consumption satisfies equation (2.11) with $r^*(t) = r(t)$ and the initial condition $B(0) = \int_{0}^{\infty} \exp(-\int_{0}^{s} r^*(u) du) (c(s) - y(s)) ds$. Given the optimal path of consumption, any solution $(\tau, t)$ consistent with (2.1), (2.17) and

$$\dot{B}(t) = r^*(t)B(t) + c(t) - y(t), \text{ for } t > 0,$$

(C.12)

is an optimal solution.

Proof. Step 1. The first part of the proof is just a direct consequence of the first-order condition $v(\tau, t) = \psi(\tau, t)$ for bond issuance. Bond prices are given by (2.4) while the government valuations are given by (2.10). Since both equations must be equal in a bounded solution, we conclude that

$$r^*(t) = r(t) = \rho - \frac{U''(c(t))}{U'(c(t))} \frac{dc}{dt},$$

15
must describe the dynamics of consumption.

Step 2. The second part of the proof derives the law of motion of \( B(t) \). First we take the derivative with respect to time at both sides of definition (2.17), that we repeat for completion:

\[
B(t) = \int_0^T \psi(\tau, t) f(\tau, t) \, d\tau.
\]

Recall that, from the law of motion of debt, equation (2.1), it holds that:

\[
\iota(\tau, t) = -\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \tau}.
\]

To express the budget constraint in terms of \( f \), we substitute \( \iota(\tau, t) \) into the budget constraint:

\[
c(t) = y(t) - f(0, t) + \int_0^T \left[ \psi(\tau, t) \left( \frac{\partial f}{\partial t} - \frac{\partial f}{\partial \tau} \right) - \delta f(\tau, t) \right] \, d\tau.
\] (C.13)

We would like to rewrite equation (C.13). Therefore, first, we apply integration by parts to the following expression:

\[
\int_0^T \psi(\tau, t) \frac{\partial f}{\partial \tau} \, d\tau = \psi(T, t)f(T, t) - \psi(0, t)f(0, t) - \int_0^T \frac{\partial \psi}{\partial \tau} f(\tau, t) \, d\tau.
\]

As long as the solution is smooth, it holds that \( f(T, t) = 0 \). Further, recall that by construction \( \psi(0, t) = 1 \). Hence:

\[
\int_0^T \psi(\tau, t) \frac{\partial f}{\partial \tau} \, d\tau = -f(0, t) - \int_0^T \frac{\partial \psi}{\partial \tau} f(\tau, t) \, d\tau.
\] (C.14)

Second, from the pricing equation of international investors, we know that

\[
\frac{\partial \psi}{\partial \tau} = -r^*(t) \psi(\tau, t) + \delta + \frac{\partial \psi}{\partial t}.
\]

Then, we obtain:

\[
\int_0^T \psi(\tau, t) \frac{\partial f}{\partial \tau} \, d\tau = -f(0, t) - \int_0^T \left[ \delta + \psi_1(\tau, t) - r(t) \psi(\tau, t) \right] f(\tau, t) \, d\tau.
\] (C.15)

We substitute (C.14) and (C.15) into (C.13), and thus:

\[
c(t) = y(t) - f(0, t) + \int_0^T \left[ \psi(\tau, t) \frac{\partial f}{\partial t} - \delta f(\tau, t) \right] \, d\tau - \left\{ -f(0, t) - \int_0^T \left[ \delta + \psi_1(\tau, t) - r(t) \psi(\tau, t) \right] f(\tau, t) \, d\tau \right\} = y(t) + \int_0^T \left[ \psi(\tau, t) f_1(\tau, t) + \psi_1(\tau, t) f(\tau, t) \right] \, d\tau - \int_0^T r^*(t) \psi(\tau, t) f(\tau, t) \, d\tau.
\]

Rearranging terms and employing the definitions above, we obtain:

\[
\dot{B}(t) = c(t) - y(t) + r^*(t)B(t),
\]

as desired.

\[\square\]
C.5 Limiting distribution: $\bar{\lambda} \to 0$

**Proposition 8.** (Limiting distribution) In the limit as liquidity costs vanish, $\bar{\lambda} \to 0$, the asymptotic optimal issuance is given by

$$ t_{\infty}^{\bar{\lambda}\to0}(\tau) = \lim_{\lambda \to 0} t_{\infty}(\tau) = \frac{1 + [\frac{r^*}{\delta} - 1] r_{ss}^* \tau e^{-r_{ss}^* \tau} \psi_{ss}(T)}{1 + [\frac{r^*}{\delta} - 1] r_{ss}^* T e^{-r_{ss}^* T} \psi_{ss}(\tau)} $$

(C.16)

where constant $\kappa > 0$ is such that $y_{ss} = f_{\infty}^{\bar{\lambda}\to0}(0) + \int_0^T \left[ t_{\infty}^{\bar{\lambda}\to0}(\tau) \frac{\psi_{ss}(\tau)}{\tau} \right] d\tau = 0$, and $f_{\infty}^{\bar{\lambda}\to0}(\tau) = \int_\tau^T t_{\infty}^{\bar{\lambda}\to0}(s)\; ds$.

**Proof.** Consider the following limit:

$$ t_{\infty}^{\bar{\lambda}\to0}(\tau) \equiv \lim_{\lambda \to 0} t_{\infty}(\tau, r_{\infty}(\bar{\lambda})) = \lim_{\lambda \to 0} \psi_{ss}(\tau) - \frac{\psi_{ss}(\tau)}{\lambda} \left[ \delta \left( 1 - e^{-r_{ss}^* \tau} \right) - \delta \left( 1 - e^{-r_{\infty}(\bar{\lambda}) \tau} \right) \right].$$

This is a limit of the form $\frac{0}{0}$ as $\lim_{\lambda \to 0} r_{\infty}(\bar{\lambda}) = r^*$. We do not have an expression for $r_{\infty}(\bar{\lambda})$, so we cannot apply L'Hôpital's rule directly. Instead, we compute:

$$ \lim_{\lambda \to 0} \frac{t_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{t_{\infty}(T, r_{\infty}(\bar{\lambda}))} = \lim_{r_{\infty}(\bar{\lambda}) \to r^*} \frac{\delta \left( 1 - e^{-r_{ss}^* \tau} \right) - \delta \left( 1 - e^{-r_{\infty}(\bar{\lambda}) \tau} \right)}{r_{\infty}(\bar{\lambda})} \left[ \frac{\psi_{ss}(\tau)}{\tau} e^{-r_{ss}^* \tau} + \frac{\psi(T)}{T} \right],$$

which also has a limit of the form $\frac{0}{0}$. Now we can apply L'Hôpital's. We obtain:

$$ \lim_{\lambda \to 0} \frac{t_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{t_{\infty}(T, r_{\infty}(\bar{\lambda}))} = \frac{-\delta r^* e^{-r_{ss}^* \tau} + \delta \left( 1 - e^{-r_{ss}^* \tau} \right)}{r^* T} + e^{-r_{ss}^* \tau} + \frac{\psi(T)}{T} = \frac{1 + [-1 + (r^* / \delta - 1) r_{ss}^* T] e^{-r_{ss}^* \tau}}{1 + [-1 + (r^* / \delta - 1) r_{ss}^* T] e^{-r_{ss}^* \tau}},$$

If we define

$$ \kappa = \lim_{\lambda \to 0} t_{\infty}(T, r_{\infty}(\bar{\lambda})) $$

then

$$ \lim_{\lambda \to 0} t_{\infty}(\tau, r_{\infty}(\bar{\lambda})) = \frac{1 + [-1 + (r^* / \delta - 1) r_{ss}^* T] e^{-r_{ss}^* \tau}}{1 + [-1 + (r^* / \delta - 1) r_{ss}^* T] e^{-r_{ss}^* \tau}} \psi(\tau).$$

The value of $\kappa$ then must be consistent with zero consumption:

$$ y_{ss} = f_{\infty}^{\bar{\lambda}\to0}(0) + \int_0^T \left[ t_{\infty}^{\bar{\lambda}\to0}(\tau) \psi_{ss}(\tau) - \delta f_{\infty}^{\bar{\lambda}\to0}(\tau) \right] d\tau = 0, $$

for $f_{\infty}^{\bar{\lambda}\to0}(\tau) = \int_\tau^T t_{\infty}^{\bar{\lambda}\to0}(s)\; ds$.  

---

We drop the sub-index ss to ease the notation.
C.6 Proof of Proposition 2

We need first the following lemma:

**Lemma 1.** Given a fixed $t$, the Gâteaux derivative of the post-shock value functional $V [f (\cdot, t)]$, defined by equation (2.6), with respect to the debt distribution $f (\cdot, t)$ is the valuation $j(\tau, t) = -U'(c(t))v(\tau, t)$ satisfying equation (2.8):

$$
\frac{d}{da} V [f (\tau, t) + ah(\tau, t)]\big|_{a=0} = \int_0^T j(\tau, t) h(\tau, t) d\tau.
$$

**Proof.** To simplify notation assume, without loss of generality, that $t = 0$. To avoid confusions, we denote by $(t^*, f^*)$ the optimal issuance policy and debt distributions. First, note that

$$
V [f (\cdot, 0)] = \mathcal{L} [t^*, f^*].
$$

This follows from the fact that

$$
V [f (\cdot, 0)] = \int_0^\infty e^{-pt} U \left( y(t) - f^*(0, t) + \int_0^T \left[ q(\tau, t, t^*) - \delta f^*(\tau, t) \right] d\tau \right) dt
\quad \quad = \int_0^\infty e^{-pt} U \left( y(t) - f^*(0, t) + \int_0^T \left[ q(\tau, t, t^*) - \delta f^*(\tau, t) \right] d\tau \right) dt
\quad \quad + \int_0^T \int_0^T e^{-pt} j^*(\tau, t) \left( \frac{\partial f^*}{\partial t} + t^*(\tau, t) + \frac{\partial f^*}{\partial \tau} \right) d\tau dt,
\quad = \mathcal{L} [t^*, f^*].
$$

The first line is the definition of $V [f (\cdot, 0)]$, the second line is the fact that

$$
-\frac{\partial f^*}{\partial t} + t^*(\tau, t) + \frac{\partial f^*}{\partial \tau} = 0
$$

for for every $\tau, t$ and the last line is the definition of the Lagrangian. Next we compute $\frac{d}{da} \mathcal{L} [t^*, f^* + ah(\tau, 0)]\big|_{a=0}$.

The derivative with respect to a general variation $h(\tau, t)$ is given by:

$$
\frac{d}{da} \mathcal{L} [t^*, f^* + ah(\tau, 0)]\big|_{a=0} = \int_0^\infty e^{-pt} U' \left( c(t) \right) \left[ -h(0, t) - \int_0^T \delta h(\tau, t) d\tau \right] dt
\quad \quad - \int_0^T \int_0^T e^{-pt} \frac{\partial h}{\partial t} j(\tau, t) d\tau dt
\quad \quad + \int_0^T \int_0^T e^{-pt} \frac{\partial h}{\partial \tau} j(\tau, t) d\tau dt.
$$

We employ integration by parts to show that:

$$
\int_0^T \int_0^T e^{-pt} \frac{\partial h}{\partial t} j(\tau, t) d\tau dt = \int_0^T \left( \lim_{s \to \infty} e^{-ps} [h(\tau, s) j(\tau, s)] - h(\tau, 0) j(\tau, 0) \right) d\tau
\quad \quad - \int_0^T \int_0^\infty e^{-pt} \left( \frac{\partial j(\tau, t)}{\partial t} - \rho j(\tau, t) \right) h(\tau, t) dt d\tau
$$

and

$$
\int_0^T e^{-pt} \int_0^T \frac{\partial h}{\partial \tau} j(\tau, t) d\tau dt = \int_0^\infty e^{-pt} \left[ j(T, t) j(T, t) - h(0, t) j(0, t) - \int_0^T \delta h(\tau, t) \frac{\partial j}{\partial \tau} d\tau \right] dt.
$$
Note that we have not yet used optimality. Using the particular case of interest, i.e.:

\[ h(\tau, 0) = h(\tau, t)\delta(t), \]

where \( \delta(t) \) is the Dirac delta, and plugging it in equation (C.17):

\[
\frac{d}{d\alpha} L [\tau^*, f^* + ah(\cdot, 0)]\bigg|_{\alpha=0} = U'(c^*(0)) \left[ -h(0, 0) - \int_0^T \delta h(\tau, 0) d\tau \right] \\
+ \int_0^T h(\tau, 0) j(\tau, 0) d\tau \\
+ \int_0^T \left( \frac{\partial j(\tau, 0)}{\partial t} - \rho j(\tau, 0) \right) h(\tau, 0) d\tau, \\
- \left. h(0, 0) j(0, 0) - \int_0^T h(\tau, 0) \frac{\partial j}{\partial \tau} d\tau \right|_{\alpha=0}.
\]

Because \((\tau^*, f^*)\) is an optimum, we know that for all \(\tau \in (0, T]\) the following holds:

\[
\frac{\partial j(\tau, 0)}{\partial t} - \rho j(\tau, 0) - \frac{\partial j(\tau, 0)}{\partial \tau} - \delta = 0,
\]

and for \(\tau = 0\) it also holds that \(U'(c^*(0)) = -j(0, 0)\). This implies that

\[
\frac{d}{d\alpha} L [\tau^*, f^* + ah(\cdot, 0)]\bigg|_{\alpha=0} = \int_0^T h(\tau, 0) j(\tau, 0) d\tau.
\]

Thus,

\[
\frac{d}{d\alpha} L [\tau^*, f^* + ah(\cdot, 0)]\bigg|_{\alpha=0} = \int_0^T h(\tau, 0) j(\tau, 0) d\tau \\
= \frac{d}{d\alpha} V [f(\cdot, 0) + ah(\cdot, 0)]\bigg|_{\alpha=0}.
\]

\[\square\]

**Proof.** Proposition 2. Then we can proceed with the proof of Proposition 2. The Lagrangian is:

\[
\mathcal{L} [\hat{\ell}, \hat{f}] = \mathbb{E}^{\hat{\rho}} \left\{ \int_0^{t^\rho} e^{-\rho s} U(\hat{\ell}(s)) ds + e^{-\rho t^\rho} \mathbb{E}^{\hat{\rho}} X \left\{ V(\hat{f}(\cdot, t^\rho), \hat{X}(t^\rho)) \right\} \right\} \\
+ \int_0^{t^\rho} \int_0^T e^{-\rho s} j(\tau, s) \left( -\frac{\partial \hat{f}}{\partial s} + i(\tau, s) + \frac{\partial \hat{f}}{\partial \tau} \right) d\tau ds
\]

where \(\mathbb{E}^{\hat{\rho}}\) denotes the expectation with respect to the random time \(t^\rho\). In this case \(j(\tau, s)\) is the Lagrange multiplier associated to the law of motion of debt, before the shock.

**Step 1. Re-writing the Lagrangian.** Proceeding as in the proof of the risk-less case, as an intermediate step we integrate by parts the terms that involve time or maturity derivatives of \(\hat{f}\). The Lagrangian \(\mathcal{L} [\hat{\ell}, \hat{f}]\) can thus be
expressed as:

\[ \mathcal{L} \left[ i, f \right] = E^\omega \left[ \int_0^T e^{-\rho s} U \left( \hat{\epsilon}(s) \right) ds + e^{-\rho \tau^0} V \left( \hat{f} \left( \cdot, \tau^0 \right), X \left( \tau^0 \right) \right) \right. \]
\[ \left. - \int_0^T e^{-\rho s} \hat{f} \left( \tau, \tau^0 \right) \hat{j} \left( \tau, \tau^0 \right) d\tau + \int_0^T \hat{f} \left( \tau, 0 \right) \hat{j} \left( \tau, 0 \right) d\tau \right. \]
\[ \left. + \int_0^T \int_0^T e^{-\rho s} \hat{f} \left( \tau, s \right) \left( \frac{\partial j}{\partial s} - \rho j \left( \tau, s \right) \right) d\tau ds \right. \]
\[ \left. + \int_0^T e^{-\rho s} \hat{f} \left( T, s \right) \hat{j} \left( T, s \right) ds - \int_0^T e^{-\rho s} \hat{f} \left( 0, s \right) \hat{j} \left( 0, s \right) ds \right. \]
\[ \left. - \int_0^T \int_0^T e^{-\rho s} \hat{f} \left( \tau, s \right) \frac{\partial j}{\partial \tau} d\tau ds \right. \]
\[ \left. + \int_0^T e^{-\rho s} \hat{f} \left( \tau, s \right) \hat{i} \left( \tau, s \right) d\tau ds. \right] \]

If we group terms, substitute the terminal conditions \( \dot{\hat{f}} \left( T, s \right) = 0 \) and compute the expected value with respect to \( \tau^0 \), we can express the Lagrangian \( \mathcal{L} \left[ i, \hat{f} \right] \) as:

\[ \mathcal{L} \left[ i, \hat{f} \right] = \int_0^\infty e^{-\left(\rho \phi^0\right)s} U \left( \hat{\epsilon}(s) \right) ds \]
\[ + \int_0^\infty e^{-\left(\rho \phi^0\right)s} \rho V \left[ \hat{f} \left( \cdot, s \right) \right] ds \]
\[ - \int_0^\infty \int_0^T e^{-\left(\rho \phi^0\right)s} \rho \hat{f} \left( \tau, s \right) \hat{j} \left( \tau, s \right) d\tau ds \]
\[ + \int_0^T \hat{f} \left( \tau, 0 \right) \hat{j} \left( \tau, 0 \right) d\tau \]
\[ + \int_0^T \int_0^T e^{-\left(\rho \phi^0\right)s} \hat{f} \left( \tau, s \right) \left( \frac{\partial j}{\partial s} - \rho j \left( \tau, s \right) \right) d\tau ds \]
\[ - \int_0^\infty \int_0^T e^{-\left(\rho \phi^0\right)s} \hat{f} \left( 0, s \right) \hat{j} \left( 0, s \right) ds \]
\[ - \int_0^\infty \int_0^T e^{-\left(\rho \phi^0\right)s} \hat{f} \left( \tau, s \right) \frac{\partial j}{\partial \tau} d\tau ds \]
\[ + \int_0^T \int_0^T e^{-\left(\rho \phi^0\right)s} \hat{f} \left( \tau, s \right) \hat{i} \left( \tau, s \right) d\tau ds. \]

**Step 2: Gâteaux derivatives.** Next, we compute the Gâteaux derivatives with respect to each of the two arguments of the Lagrangian at a time. **Step 2.1: Gâteaux derivative with respect to issuances.** We consider a perturbation around optimal issuances. Equalizing the Gâteaux derivative with respect to issuances to zero, i.e. \( \mathcal{L} \left[ i + ah, \hat{f} \right] \vert_{a=0} = 0 \), the result is identical to the riskless case:

\[ U' \left( \hat{\epsilon} \left( t \right) \right) \left( \frac{\partial q}{\partial i} \hat{i} \left( t, t \right) + q \left( t, t, \hat{i} \right) \right) = -\hat{j} \left( t, t \right). \]

**Step 2.2: Gateaux derivative of V with respect to the debt density.** The Gâteaux derivative of the continuation value with respect to the debt density is

\[ \frac{d}{da} \bigg|_{a=0} V \left[ \hat{f} \left( \cdot, s \right) + ah \left( \cdot, s \right), X \left( s \right) \right] = E^X \left\{ \int_0^T \hat{j} \left( \tau, s \right) h \left( \tau, s \right) d\tau \right\}. \]
where we have applied Lemma 1. \textbf{Step 2.3: Gateaux derivative of the Lagrangian with respect to the debt density.}

Since the distribution at the beginning $\hat{f}(\tau,0)$ is given, any feasible perturbation must feature $h(\tau,0) = 0$ for any $\tau \in (0,T]$. In addition, we know that $h(T,t) = 0$, because $\hat{f}(T,t) = 0$. The Gateaux derivative of the Lagrangian with respect to the debt density is:

$$
\frac{d}{da} \mathcal{L} \left[ \hat{f}, \hat{f} + ah \right]_{a=0} = \int_0^\infty e^{-\rho \phi s} U'(\hat{\epsilon}(s)) \left[ -h(0,s) + \int_0^T (-\delta) h(\tau,s) \, d\tau \right] ds
$$

$$
+ \int_0^\infty \int_0^T e^{-\rho \phi s} \phi \mathbb{E}_s X \left[ j(\tau,s) h(\tau,s) \right] d\tau ds
$$

$$
- \int_0^\infty \int_0^T e^{-\rho \phi s} \theta h(\tau,s) \hat{j}(\tau,s) d\tau ds
$$

$$
+ \int_0^T h(\tau,0) \hat{j}(\tau,0) d\tau
$$

$$
+ \int_0^\infty \int_0^T e^{-\rho \phi s} h(\tau,s) \left( \frac{\partial j}{\partial s} - \rho j(\tau,s) \right) ds d\tau
$$

$$
- \int_0^\infty e^{-\rho \phi s} h(0,s) \hat{j}(0,s) ds
$$

$$
- \int_0^\infty \int_0^T e^{-\rho \phi s} h(\tau,s) \frac{\partial j}{\partial \tau} d\tau ds.
$$

The value of the Gateaux derivative of the Lagrangian for any perturbation must be zero. Thus, again a necessary condition is to have all terms that multiply any entry of $h(\tau,s)$ add up to zero. We summarize the necessary conditions into:

$$
\rho j(\tau,s) = -\delta U'(\hat{\epsilon}(s)) + \frac{\partial j}{\partial s} - \frac{\partial j}{\partial \tau} + \phi \left[ \mathbb{E}_s X j(\tau,s) - j(\tau,s) \right], \quad (C.18)
$$

$$
\hat{j}(0,s) = -U'(\hat{\epsilon}(s)). \quad (C.19)
$$

\textbf{Step 3: From Lagrange multipliers to valuations.} We now employ the definitions of $\hat{v}(\tau,s) = -\hat{j}(\tau,s) / U'(\hat{\epsilon}(s))$ and $v(\tau,s) = -j(\tau,s) / U'(c(s))$. Thus, we can express equations (C.18)-(C.19) as:

$$
\hat{r}(s) \hat{v}(\tau,s) = \delta + \frac{\partial \hat{v}}{\partial s} - \frac{\partial \hat{v}}{\partial \tau} + \phi \left[ \mathbb{E}_s X \frac{U'(c(s))}{U'(\hat{\epsilon}(s))} v(\tau,s) - \hat{v}(\tau,s) \right]
$$

$$
\hat{v}(0,s) = 1.
$$

Therefore valuations can be expressed as:

$$
\hat{v}(\tau,t) = e^{-\int_0^t \hat{r}(s) + \phi} ds
$$

$$
+ \theta \int_t^{t+\tau} e^{-\int_0^u \hat{r}(s) + \phi} du \left( \delta + \mathbb{E}_s X \left[ \frac{U'(c(t+s))}{U'(\hat{\epsilon}(t+s))} v(\tau-s,t+s) \right] \right) ds,
$$

as we wanted to show. \hfill \Box
C.7 Proof of Proposition 3

Proof. Step 1. Setting the Lagrangian. Let $V \left[ \hat{f} \left( \cdot, t^o \right), X \left( t^o \right) \right]$ denote the expected value of the government, at the instant $t^o$ where the option to default is available, but prior to the decision of default. This value equals:

$$
E_{t^o} \left[ \begin{array}{cc}
\Gamma \left( V \left[ \hat{f} \left( \cdot, t^o \right), X \left( t^o \right) \right], X \left( t^o \right) \right) + \Theta \left( V \left[ \hat{f} \left( \cdot, t^o \right), X \left( t^o \right) \right] V \left[ \hat{f} \left( \cdot, t^o \right), X \left( t^o \right) \right] \right)
\end{array} \right],
$$

where the first term in the expectation is the expected utility conditional on default given by $\Gamma (x) \equiv \int\!_x^\infty \! x \! d \! \Theta (z)$. The second term is the probability of no default time the perfect-foresight value. The Lagrangian is:

$$
\mathcal{L} \left[ \hat{\iota}, \hat{f}, \hat{\psi} \right] =
E_{t^o} \left[ \int_{t^o}^{T} e^{-\rho s} U \left( \hat{c}(s) \right) ds + e^{-\rho t^o} V \left[ \hat{f} \left( \cdot, t^o \right), X \left( t^o \right) \right] 
\right. \\
+ \int_{t^o}^{T} \int_{t^o}^{T} e^{-\rho s} \hat{j} (\tau, s) \left( - \frac{\partial \hat{f}}{\partial s} + \hat{i} (\tau, s) + \frac{\partial \hat{f}}{\partial \tau} \right) d \tau ds \\
\left. + \int_{t^o}^{T} \int_{t^o}^{T} e^{-\rho s} \hat{\mu} (\tau, s) \left( - \hat{\psi} (s) \hat{\psi} (\tau, s) + \delta + \frac{\partial \hat{\psi}}{\partial s} - \frac{\partial \hat{\psi}}{\partial \tau} \right) d \tau ds \right].
$$

In the Lagrangian, $E_{t^o}$ denotes the conditional expectation with respect to the random time $t^o$. Here $j (\tau, s)$ and $\hat{\mu} (\tau, s)$ are the Lagrange multipliers. The first set of multipliers, $j (\tau, s)$, are associated with the law of motion of debt and appears also in previous sections. The second set of multipliers, $\hat{\mu} (\tau, s)$, are associated with the law of motion of bond prices. These terms appear because the government understands how its influence on the maturity profile affects the incentives to default, and hence impacts bond prices. This happens through the terminal condition:

$$
\hat{\psi} (\tau, t^o) =
E_{t^o} \left\{ \Theta \left( V \left[ \hat{f} \left( \cdot, t^o \right), X \left( t^o \right) \right] \right) \psi (\tau, t^o) \right\},
$$

$$
\hat{\psi} (0, t) =
1.
$$

The terminal condition reflects that, at date $t^o$, the bond price is zero if default occurs. Otherwise it equals the perfect-foresight price, $\psi (\tau, t^o)$, if default does not occur.

Step 1.2. Re-writing the Lagrangian. Proceeding as in the proof of the riskless case, as an intermediate step we integrate by parts the terms that involve time or maturity derivatives of $\hat{f}$ and $\hat{\psi}$. The Lagrangian $\mathcal{L} \left[ \hat{\iota}, \hat{f}, \hat{\psi} \right]$ can
Step 1.3. Computing expectations. If we group terms, substitute the terminal conditions \( f(T, s) = 0 \) and \( \hat{\psi}(\tau, t^o) = \Theta \left( V \left[ \hat{f} (\cdot, t^o), X (t^o) \right] \right) \psi(\tau, t^o) \) and compute the expected value with respect to \( t^o \), we can express
the Lagrangian $\mathcal{L} [\hat{t}, \hat{f}, \hat{\psi}]$ as:

$$\int_0^\infty e^{-(\rho + \phi)s} U (\hat{e}(s)) \, ds$$

$$- \int_0^\infty e^{-(\rho + \phi)s} \hat{f} (0, s) \hat{j} (0, s) \, ds$$

$$- \int_0^\infty e^{-(\rho + \phi)s} \left[ \hat{\mu} (T, s) \hat{\psi} (T, s) - \mu (0, s) \hat{\psi} (0, s) \right] \, ds$$

$$+ \int_0^\infty \int_0^T e^{-(\rho + \phi)s} \hat{f} (\tau, s) \left( \frac{\partial \hat{j}}{\partial s} - \rho \hat{j} (\tau, s) \right) ds \, d\tau$$

$$- \int_0^\infty \int_0^T e^{-(\rho + \phi)s} \hat{j} (\tau, s) \frac{\partial \hat{j}}{\partial \tau} d\tau ds$$

$$+ \int_0^\infty \int_0^T e^{-(\rho + \phi)s} \hat{\mu} (\tau, s) \left( -\hat{\rho} (s) \hat{\psi} (\tau, s) + \delta \right) d\tau ds$$

$$- \int_0^\infty \int_0^T e^{-(\rho + \phi)s} \hat{\psi} (\tau, s) \left( \frac{\partial \hat{\mu}}{\partial s} - \rho \hat{\mu} (\tau, s) \right) d\tau ds$$

$$+ \int_0^\infty \int_0^T e^{-(\rho + \phi)s} \hat{\psi} (\tau, s) \frac{\partial \hat{\mu}}{\partial \tau} d\tau ds$$

$$+ \int_0^T f (\tau, 0) \hat{j} (\tau, 0) \, d\tau - \int_0^T \hat{\mu} (\tau, 0) \hat{\psi} (\tau, 0) \, d\tau$$

$$+ \int_0^\infty e^{-(\rho + \phi)s} \phi \mathcal{V} \left[ \hat{f} (\cdot, s), X(s) \right] \, ds$$

$$- \int_0^\infty \int_0^T e^{-(\rho + \phi)s} \phi \hat{\mu} (\tau, s) \hat{j} (\tau, s) \, d\tau ds$$

$$+ \int_0^\infty \int_0^T e^{-(\rho + \phi)s} \phi \hat{\psi} (\tau, s) \mathbb{E}_s^X \left\{ \theta \left( \mathcal{V} \left[ \hat{f} (\cdot, s), X(s) \right] \right) \psi (\tau, s) \right\} \, d\tau ds.$$

Next, we compute the Gâteaux derivatives with respect to each of the three arguments of the value function at a time.

**Step 2. Computing the derivatives.**

**Step 2.1. Gâteaux derivative with respect to the issuances.** If we consider a perturbation around issuances and equalize it to zero, $\frac{d}{da} \mathcal{L} [\hat{t} + ah, \hat{f}, \hat{\psi}] \big|_{a=0} = 0$, the result is identical to the risk-less case:

$$\mathcal{U}' (\hat{e} (t)) \left( \frac{d}{dt} \hat{j} (\tau, t) + q (t, \tau, h) \right) = -\hat{j} (\tau, t).$$

**Step 2.2. Gâteaux derivative with respect to the debt density.** Since the distribution at the beginning $f (\tau, 0)$ is given, any feasible perturbation must feature $h (\tau, 0) = 0$ for any $\tau \in (0, T]$. In addition, we know that $h(T, t) = 0$, because $f(T, t) = 0$. The Gâteaux derivative of the continuation value with respect to the debt density is:

$$\frac{d}{da} \mathcal{V} \left[ \hat{f} (\cdot, s) + ah (\cdot, s), X(s) \right] \big|_{a=0} = \mathbb{E}_s^X \left\{ \Theta \left( \mathcal{V} \left[ \hat{f} (\cdot, s), X(s) \right] \right) \int_0^T \hat{j} (\tau, s) h (\tau, s) \, d\tau \right\},$$

where we have taken into account the fact that $\frac{d}{dx} (\Gamma (x) + \Theta (x) x) = \Theta (x)$ and from the perfect foresight problem—:

$$\frac{d}{da} \mathcal{V} \left[ \hat{f} (\cdot, s) + ah (\cdot, s) \right] \big|_{a=0} = \int_0^T \hat{j} (\tau, s) h (\tau, s) \, d\tau.$$
Similarly, the Gâteaux derivative of the terminal bond price with respect to the debt density is

\[
\frac{d}{d\alpha} \mathbb{E}_s^X \left\{ \Theta \left( V \left[ f \left( \cdot, s \right) + ah \left( \cdot, s \right), X \left( s \right) \right] \right) \psi \left( \tau, s \right) \right\} \bigg|_{\alpha = 0} = \ldots
\]

\[
\mathbb{E}_s^X \left\{ \theta \left( V \left[ \hat{f} \left( \cdot, s \right), X \left( s \right) \right] \right) \psi \left( \tau, s \right) \int_0^T j \left( \tau', s \right) h \left( \tau', s \right) d\tau' \right\},
\]

where \( \theta(x) \equiv \frac{d}{dx} \Theta (x) \) is the probability density. The Gâteaux derivative of the Lagrangian with respect to the debt density, \( \frac{d}{d\alpha} \mathcal{L} \left[ i, \hat{f} + ah, \hat{\psi} \right] \bigg|_{\alpha = 0} \), is thus:

\[
\int_0^\infty e^{-\left(\rho + \phi\right) s} U' \left( \hat{\epsilon} \left( s \right) \right) \left[ -h \left( 0, s \right) + \int_0^T \left(-\delta \right) h \left( \tau, s \right) d\tau \right] ds
\]

\[- \int_0^\infty e^{-\left(\rho + \phi\right) s} h \left( 0, s \right) j \left( 0, s \right) ds
\]

\[+ \int_0^\infty \int_0^T e^{-\left(\rho + \phi\right) s} h \left( \tau, s \right) \left( \frac{\partial j}{\partial s} - \rho j \left( \tau, s \right) \right) ds d\tau
\]

\[- \int_0^\infty \int_0^T e^{-\left(\rho + \phi\right) s} h \left( \tau, s \right) \frac{\partial j}{\partial \tau} d\tau ds
\]

\[+ \int_0^T h \left( \tau, 0 \right) j \left( \tau, 0 \right) d\tau
\]

\[+ \int_0^\infty \int_0^T e^{-\left(\rho + \phi\right) s} \theta \mathbb{E}_s^X \left\{ \Theta \left( V \left[ f \left( \cdot, s \right), X \left( s \right) \right] \right) j \left( \tau, s \right) \right\} h \left( \tau, s \right) d\tau ds
\]

\[- \int_0^\infty \int_0^T e^{-\left(\rho + \phi\right) s} \phi h \left( \tau, s \right) j \left( \tau, s \right) d\tau ds
\]

\[+ \int_0^\infty \int_0^T e^{-\left(\rho + \phi\right) s} \phi \mu \left( m, s \right) \mathbb{E}_s^X \left\{ \Theta \left( V \left[ f \left( \cdot, s \right), X \left( s \right) \right] \right) \psi \left( m, s \right) \int_0^T j \left( \tau, s \right) h \left( \tau, s \right) d\tau \right\} dmds.
\]

The value of the Gâteaux derivative of the Lagrangian for any perturbation, must be zero, i.e.

\[
\frac{d}{d\alpha} \mathcal{L} \left[ i, \hat{f} + ah, \hat{\psi} \right] \bigg|_{\alpha = 0} = 0.
\]

Thus, a necessary condition is that all terms that multiply any entry of \( h \left( \tau, s \right) \) add up to zero. We summarize the necessary conditions into:

\[
\rho \hat{j} \left( \tau, s \right) = \left(-\delta \right) U' \left( \hat{\epsilon} \left( s \right) \right) + \frac{\partial \hat{j}}{\partial s} - \frac{\partial j}{\partial \tau}
\]

\[+ \phi \mathbb{E}_s^X \left\{ \left[ \Theta \left( V \left[ f \left( \cdot, s \right), X \left( s \right) \right] \right) + \theta \left( V \left[ f \left( \cdot, s \right), X \left( s \right) \right] \right) \int_0^T \mu \left( m, s \right) \psi \left( m, s \right) dm \right] j \left( \tau, s \right) \right\}
\]

\[j \left( 0, s \right) = -U' \left( \hat{\epsilon} \left( s \right) \right).
\]

**Step 2.3. Gâteaux derivative with respect to the bond price.** In the case of the Gâteaux derivatives with respect to the evolution of the price \( \hat{\psi}, \frac{d}{d\alpha} \mathcal{L} \left[ i, f, \hat{\psi} + ah \right] \bigg|_{\alpha = 0} \), we need to work first with the Lagrangian before expectations have been computed. The reason is the following: only bonds that mature after default can be affected by the Government’s policies and hence the variations have to be zero for those bonds that mature before default, \( \tilde{h} \left( \tau, t \right) = 0 \), if \( \tau + t < t^0 \). To incorporate this, we assume that admissible perturbations are of the form \( \tilde{h} \left( \tau, t \right) = h \left( \tau, t \right) 1_{ \left\{ \tau + t \geq t^0 \right\} } \).
where \( h(\tau, t) \) is unrestricted. The Gâteaux derivative is then

\[
\mathbb{E}^\rho \left[ \int_0^T e^{-\rho s} U'(\hat{\psi}(s)) \left( \int_0^T \hat{i}(\tau, s) \frac{\partial q}{\partial \psi} 1_{\{\tau + s \geq t^\rho\}} h(\tau, s) d\tau \right) ds \right] \\
+ \int_0^T \int_0^T e^{-\rho s} \hat{\mu}(\tau, t) \left( -\hat{r}(s) h(\tau, s) \right) d\tau ds \\
- \int_0^T \hat{\mu}(\tau, 0) 1_{\{\tau \geq t^\rho\}} h(\tau, 0) d\tau \\
- \int_0^T \int_0^T e^{-\rho s} 1_{\{\tau + s \geq t^\rho\}} h(\tau, s) \left( \frac{\partial \hat{\mu}}{\partial s} - \rho \hat{\mu}(\tau, s) \right) d\tau ds \\
- \int_0^T e^{-\rho s} \left[ 1_{\{T + s \geq t^\rho\}} h(T, s) \hat{\psi}(T, s) - e^{-\rho s} 1_{\{s \geq t^\rho\}} h(0, s) \hat{\psi}(0, s) \right] ds \\
+ \int_0^T \int_0^T e^{-\rho s} 1_{\{\tau + s \geq t^\rho\}} h(\tau, s) \frac{\partial \hat{\mu}}{\partial \tau} d\tau ds,
\]

Note that the perturbation is only around \( \hat{\psi}(\tau, s) \) and not \( \psi(\tau, s) \), the terminal price after default, which is given. Since at maturity, bonds have a value of 1, \( h(0, s) = 0 \), because no perturbation can affect that price. If we compute the expectation with respect to the random arrival time, \( t^\rho \), we get:

\[
\int_0^\infty e^{-(\rho + \phi)s} \left( 1 - e^{-\phi T} \right) U'(\hat{\psi}(s)) \left[ \int_0^T \hat{i}(\tau, s) \frac{\partial q}{\partial \psi} h(\tau, s) d\tau \right] ds \\
+ \int_0^\infty \int_0^T e^{-(\rho + \phi)s} \left( 1 - e^{-\phi T} \right) \hat{\mu}(\tau, s) \left( -\hat{r}(s) h(\tau, s) \right) d\tau ds \\
- \int_0^T \left( 1 - e^{-\phi T} \right) \mu(\tau, 0) h(\tau, 0) d\tau \\
- \int_0^\infty \int_0^T e^{-(\rho + \phi)s} \left( 1 - e^{-\phi T} \right) h(\tau, s) \left( \frac{\partial \hat{\mu}}{\partial s} - \rho \hat{\mu}(\tau, s) \right) d\tau ds \\
- \int_0^\infty e^{-(\rho + \phi)s} \left( 1 - e^{-\phi T} \right) \mu(T, s) h(T, s) ds \\
+ \int_0^\infty \int_0^T e^{-(\rho + \phi)s} \left( 1 - e^{-\phi T} \right) \hat{\psi}(\tau, s) \frac{\partial \hat{\mu}}{\partial \tau} d\tau ds,
\]

where we use

\[
\mathbb{E}^\rho \left[ 1_{\{T + s \geq t^\rho\}} \right] = e^{-\phi s} \left( 1 - e^{-\phi T} \right).
\]

Again, as the Gâteaux derivative should be zero for any suitable \( h(\tau, s) \), the optimality condition is

\[
(\hat{r}(s) - \rho) \hat{\mu}(\tau, s) = U'(\hat{\psi}(s)) \hat{i}(\tau, s) \frac{\partial q}{\partial \psi} - \frac{\partial \hat{\mu}}{\partial s} + \frac{\partial \hat{\mu}}{\partial \tau},
\]

\[
\hat{\mu}(T, s) = 0, \\
\hat{\mu}(\tau, 0) = 0.
\]

The solution to this PDE is

\[
\hat{\mu}(\tau, s) = \int_{\max\{s + \tau - T, 0\}}^s e^{-\int_s^\tau (\rho(u) - \rho) du} U'(\hat{\psi}(z)) \hat{i}(\tau + s - z, z) \frac{\partial q}{\partial \psi}(\tau + s - z, z) dz.
\]
If we integrate the discount factor of the government with respect to time, we obtain the following identity:

$$\int_{z}^{s} \hat{\rho} (u) \, du = \int_{z}^{s} \rho \, du - \int_{z}^{s} \frac{U''(\hat{c}(u)) \hat{c}'(u)}{U'(\hat{c}(u))} \hat{c}'(u) \, du.$$ 

Therefore, we have that

$$\int_{z}^{s} \hat{\rho} (u) \, du - \int_{z}^{s} \rho \, du = - \log \left( \frac{U' (\hat{c}(u))}{U'(\hat{c}(z))} \right).$$

We obtain the following identity:

$$e^{\int_{z}^{s} \hat{\rho} (u) \, du} = e^{\int_{z}^{s} \hat{\rho} (u) \, du} \frac{U'(\hat{c}(s))}{U'(\hat{c}(z))}.$$ 

Thus, the PDE for $\hat{\mu} (\tau, s)$ can be written as

$$\hat{\mu} (\tau, s) = U'(\hat{c}(s)) \int_{\max\{s+\tau-T,0\}}^{s} e^{-\int_{\tau}^{\sigma} (\hat{\rho}'(u) - \hat{\rho}(u)) \, du} \hat{\mu} (\tau + s - z, z) \frac{\partial \hat{q}}{\partial \phi} (\tau + s - z, z) \, dz.$$ 

Notice that

$$\hat{\mu} (\tau, s) = U'(\hat{c}(s)) \int_{\max\{s+\tau-T,0\}}^{s} e^{-\int_{\tau}^{\sigma} (\hat{\rho}'(u) - \hat{\rho}(u)) \, du} \hat{\mu} (\tau + s - z, z) \frac{\partial \hat{q}}{\partial \phi} (\tau + s - z, z) \, dz.$$

We employ this solution in the main text.

_Step 3: From Lagrange multipliers to valuations._ We now employ the definitions of $\hat{\phi} (\tau, s) = -j (\tau, s) / U'(\hat{c}(s))$ and $v (\tau, s) = -j (\tau, s) / U'(c(s))$, we can express equations (C.20)-(C.21) as

$$\hat{\rho} (\tau, s) \hat{\phi} (\tau, s) = \delta + \frac{\partial \hat{\phi}}{\partial s} - \frac{\partial \hat{\phi}}{\partial \tau} + \phi \mathbb{E}_{s}^{X} \left\{ \Theta (V(s)) + \theta (V(s)) \int_{0}^{T} \hat{\mu} (m, s) \psi (m, s) \, dm \right\} \frac{U'(c(s))}{U'(\hat{c}(s))} \nu (\tau, s) - \hat{\phi} (\tau, s),$$

$$\hat{\phi} (0, s) = 1,$$

where we use the notation $\Theta (V(s)) = \Theta (V [f (\cdot, s), X(s)])$ and $\theta (V(s)) = \theta (V [f (\cdot, s), X(s)])$. Therefore valuations can be expressed as

$$\hat{\phi} (\tau, t) = e^{-\int_{\tau}^{\tau+T} (\hat{\rho}(s)+\delta) \, ds} + \phi \int_{\tau}^{\tau+T} e^{-\int_{\tau}^{s} (\hat{\rho}(u)+\delta) \, du} \left( \delta + \mathbb{E}_{s}^{X} \left[ \Theta (V(t+s)) + \Omega(t+s) \right] \frac{U'(c(t+s))}{U'(\hat{c}(t+s))} \nu (\tau-s, t+s) \right) \, ds,$$

where

$$\Omega(t) = \theta (V(t)) \int_{0}^{T} \mu (m, t) \psi (m, t) \, dm.$$
C.8 The case of default without liquidity costs: $\bar{\lambda} = 0$

We show here that the maturity structure is indetermined in the case without liquidity costs and a finite support of $G$. In proposition 9 below we show how, if distribution $\hat{j}^o$ is a solutions of Problem (4.3), then another distribution $\hat{j}'$ is also a solution provided that

$$\int_0^T (\psi (\tau, t, X(t)) - \hat{\psi} (\tau, t)) \left( \hat{j}^o (\tau, t) - \hat{j}' (\tau, t) \right) d\tau = 0, \quad (C.22)$$

and

$$\int_0^T \left( \mathbb{E}_X \left[ \Theta \left( V \left[ \hat{j}^o (\cdot, t), X(t) \right] \right) \right] - \hat{\psi} (\tau, t) \right) \left( \hat{j}^o (\tau, t) - \hat{j}' (\tau, t) \right) d\tau = 0. \quad (C.23)$$

Consider first the case of an income shock. Here $X(t)$ does not jump after the shock arrives. If the government decides to default then the maturity profile at the moment of default is irrelevant. If the government decides instead to repay, the post-shock yield curve will be $\psi (\tau, t^o)$, which differs from $\hat{\psi} (\tau, t^o)$ as the post-shock default premium is zero. The maturity structure is indeterminate because conditions (C.22) and (C.23) are two integral equations with a continuum of unknowns, $\hat{j}' (\tau, t^o)$.

Consider next the case of an interest rate shock, in which $X(t)$ jumps with the option to default. Condition (C.22) is a system of integral equations, indexed by $X(t^o)$, where $\hat{j}' (\cdot, t^o)$ is the unknown. Provided that $X(t^o)$ may take $N$ possible values, then we have at most $N$ equations that need to be satisfied by the debt distribution. In addition we have equation (C.23) and the condition that the market debt should coincide. Notice that the number can be less than $N + 2$ as in some states the government may default and then condition C.22 is trivially satisfied for any debt profile that replicates the total debt at market prices before the shock arrival. In any case, the maturity structure is indeterminate.

The indeterminacy of the debt distribution in our model complements previous results in the literature. In particular, Aguiar et al. (Forthcoming) study a model of sovereign default similar to the one presented with the key difference that in their model the government cannot commit to future debt issuances whereas in our paper it ca, conditional on repayment. Aguiar et al. (Forthcoming) find how in that case the government only operates in the short end of the curve, making payments and retiring long-term bonds as they mature but never actively issuing or buying back such bonds. This is because short term bonds cannot be diluted. The authors also conjecture that the maturity structure would be indeterminate if the government had full commitment over its issuance path. This is precisely the case we study here, confirming their conjecture.

**Proposition 9.** Let $\{\hat{t}^o (\tau, t), \hat{j}^* (\tau, t), \hat{c}^* (t) \}_{t \in [0, t^o]}$ and $\{\hat{t}^* (\tau, t), f^* (\tau, t), c^* (t) \}_{t \in (t^o, \infty)}$ be the solution of Problem (4.3) when $\lambda (t, \tau, t) = 0$. Let $\{\hat{t}' (\tau, t), \hat{j}' (\tau, t), \hat{c}' (t) \}_{t \in [0, t^o]}$ and $\{\hat{t}^* (\tau, t), f^* (\tau, t), c^* (t) \}_{t \in (t^o, \infty)}$ be such that, for every $t \leq t^o$ and every value of $X(t)$,

$$\hat{B}^* (t) = \hat{B}' (t) \quad (C.24)$$

$$\int_0^T (\psi (\tau, t, X(t)) - \hat{\psi} (\tau, t)) \left( \hat{j}^* (\tau, t) - \hat{j}' (\tau, t) \right) d\tau = 0, \quad (C.25)$$

and

$$\int_0^T \left( \mathbb{E}_X \left[ \Theta \left( V \left[ \hat{j}^* (\cdot, t), X(t) \right] \right) \right] - \hat{\psi} (\tau, t) \right) \left( \hat{j}^* (\tau, t) - \hat{j}' (\tau, t) \right) d\tau = 0. \quad (C.26)$$

and $B^* (t) = B' (t)$ for every $t > t^o$. Then, $\hat{c}' (t) = \hat{c}^* (t)$ and $c' (t) = c^* (t)$. Thus, $\{\hat{t}' (\tau, t), \hat{j}' (\tau, t), \hat{c}' (t) \}_{t \in [0, t^o]}$ and $\{\hat{t}^* (\tau, t), f^* (\tau, t), c^* (t) \}_{t \in (t^o, \infty)}$ is also optimal.
Proof. Step 0. Default values. The value function of a policy given an initial debt $f(\cdot,0)$ is given by:

$$
\hat{V}[f(\cdot,0)] = \mathbb{E}_0\left[ \int^T_0 e^{-\rho t} U(\hat{\epsilon}(t)) \, dt + \mathbb{E}_{\mathcal{V}_t, X(t^\circ)} \left[e^{-\rho t^\circ} V^D(t^\circ, f(\cdot, t^\circ)) \right] \right]
$$

where the post-default value $V^O \left[V^D(t^\circ, f(\cdot, t^\circ), X(t^\circ)) \right] \equiv \max \{ V^D(t^\circ), V[f(\cdot, t^\circ), X(t^\circ)] \}$ and $V[f(\cdot, t^\circ), X(t^\circ)]$ is the value of the perfect-foresight solution. Note that, from the solution of the problem with perfect foresight, the value function only depends on the aggregate market value of total debt, $V[f(\cdot, t^\circ), X(t^\circ)] = V(B(t^\circ, X(t^\circ)), X(t^\circ))$, where $B(t^\circ, X(t^\circ))$ is defined as the market value of debt

$$
B(t^\circ, X(t^\circ)) = \int^T_0 \psi(\tau, t^\circ, X(t^\circ)) f(\tau, t^\circ) \, d\tau.
$$

Therefore, the post-default value

$$
V^O \left[V^D(t^\circ, f(\cdot, t^\circ), X(t^\circ)) \right] = V^O \left(V^D(t^\circ, B(t^\circ, X(t^\circ)), X(t^\circ)) \right),
$$

also only depends on the aggregate market value of total debt, $B(t^\circ, X(t^\circ))$. Because $B'(t^\circ, X(t^\circ)) = B^*(t^\circ, X(t^\circ))$ for every realization of $X(t^\circ)$ the default decision depends only on the market value of debt when the country receives the opportunity to default and not on the debt-maturity profile. Thus, continuation values are equal and it is enough to show that $\hat{\epsilon}^* (t) = \hat{\epsilon}' (t)$ for $t \leq t^\circ$ to prove that the two policies yield the same utility.

Step 1. Pre-shock prices are equal. Pre-shock prices solve

$$
\hat{f}^* (t) \hat{\psi}(\tau, t) = \delta + \frac{\partial \hat{\psi}}{\partial t} - \delta \frac{\partial \hat{\psi}}{\partial \tau} + \hat{\phi} \mathbb{E}_t^X \left[ \Theta \left(V[f(\cdot, t), X(t)]\right) \psi(\tau, t, X(t)) - \hat{\psi}(\tau, t) \right], \quad t < t^\circ
$$

$$
\hat{\psi}(\tau, t^\circ) = \mathbb{E}_t^X \left[ \Theta \left(V[f(\cdot, t^\circ), X(t^\circ)]\right) \psi(\tau, t^\circ) \right]
$$

$$
\hat{\psi}(0, t) = 1.
$$

It holds that

$$
\Theta \left(V[f^*(\cdot, t^\circ), X(t^\circ)]\right) = \Theta \left(V(B^*(t^\circ, X(t^\circ)), X(t^\circ))\right) = \Theta \left(V(B'(t^\circ, X(t^\circ)), X(t^\circ))\right).
$$

This is a consequence of the fact that $V(B^*(t^\circ, X(t^\circ)), X(t^\circ)) = V(B'(t^\circ, X(t^\circ)), X(t^\circ))$. Thus, pre-shock prices are equal for both policies.

Step 2. Law of motion of debt before the shock arrival. By definition $\hat{B}(t) = \int_0^T \hat{\psi}(\tau, t) \hat{f}(\tau, t) \, d\tau$. The dynamics of $\hat{B}(t)$ for $t < t^\circ$ are:

$$
d\hat{B}(t) = \left( \int_0^T \left( \hat{\psi}_t(\tau, t) \hat{f}(\tau, t) + \hat{\psi}(\tau, t) \hat{f}_t(\tau, t) \right) \, d\tau \right) \, dt,
$$

which, with similar derivations as in 7, yields to

$$
d\hat{B}(t) = \left( \hat{\epsilon}(t) - y(t) + \hat{f}^*(t) \hat{B}(t) + \hat{\phi} \int_0^T \left( \mathbb{E}_X \left[ \Theta \left(V(B(t, X(t)), X(t))\right) \psi(\tau, t, X(t)) \right] - \hat{\psi}(\tau, t) \right) \hat{f}(\tau, t) \, d\tau \right) \, dt.
$$

Step 3. The expected jump. Note that (C.26) for all $X_t$ implies that:

$$
\hat{\phi} \int_0^T \left( \mathbb{E}_X \left[ \Theta \left(V(B(t, X(t)), X(t))\right) \psi(\tau, t, X(t)) \right] - \hat{\psi}(\tau, t) \right) \left( \hat{f}(\tau, t) - \hat{f}^*(\tau, t) \right) \, d\tau = 0,
$$

(C.27)
Combining this equation with the law of motion of debt we get that before the shock arrival, $t < t^o$,
\[
d dB^*(t) = dB'(t).
\] (C.28)

**Step 4. The actual jump.** Condition (C.25) guarantees that the jump is the same for any $X(t)$ if the country does not default. If it defaults, condition (C.25) is trivially satisfied as $\hat{B}^*(t) = B'(t)$ and the jump is also the same as market debt is then zero. Hence $\hat{B}^*(t^o) = B'(t^o)$. Finally, taking all these results together we conclude that: $\hat{c}^*(t) = \hat{c}'(t)$ for all $t \leq t^o$. As the policy $\{i'(t, t), f'(t, t), \hat{c}'(t)\}_{t \in [0, t^o]}$ and $\{i^*(t, t), f^*(t, t), c^*(t)\}_{t \in [t^o, \infty)}$ achieves the same consumption path as the optimal, it is thus optimal. \qed

## D Calibration notes

In this Appendix, we describe the sources of the data and the calibration procedure for the parameter values and shocks used in the numerical exercises in subsection 2.6, subsection 3.3, and subsection 4.3.

### Income Process: $\rho_y$

We obtain the series for the Spanish Gross Domestic Product for the period Q1:1995 to Q2:2018 from FRED economic data [https://fred.stlouisfed.org/series/CLVMNACSCAB1GQES](https://fred.stlouisfed.org/series/CLVMNACSCAB1GQES). We estimate an AR(1) process for the detrended seasonally adjusted output. The detrending uses a Hodrick-Prescott filter with a parameter of 1600. The estimated model is $\log y_t = \rho_y \log y_{t-1} + \sigma_y \epsilon^y_{t-1}$. The estimated persistence of quarterly income ($\rho_y$) is 0.95 with a standard deviation ($\sigma_y$) of 0.375. This corresponds to a value of $\alpha_y = (1 - \rho_y)/ (3 \times \Delta t) = 0.2$, where we fix $\Delta t = 1/12$ for all our numerical exercises. The details of the numerical procedure to solve the model are described in Appendix F.

### Interest Rates Process: $\rho_r$

We obtain from the Bundesbank monthly data for the 1 month Euribor nominal rate. The period is Q1 1999 to Q2 2018. The source of the data is: [https://www.bundesbank.de/action/en/744770/bbkstatisticsearch?query=euribor](https://www.bundesbank.de/action/en/744770/bbkstatisticsearch?query=euribor). We then obtain the annualized average (geometric mean) quarterly rate, $i^q_t$, as $\left[ \prod_{t=1}^3 (1 + i^q_t) \right]^{1/3} = (1 + i^q_t)$. Using the quarterly inflation rate for the Eurozone (overall index not seasonally adjusted), obtained from the ECB Statistical Data Warehouse, the source of the data is [https://sdw.ecb.europa.eu/](https://sdw.ecb.europa.eu/), we compute the annualized real rate at a quarterly frequency as $1 + \frac{i^q_t}{1 + \pi^q_t} = 1 + r^q_t$. We then fit an AR(1) process for the level of the real interest rate $r_t = \mu + \rho r_{t-1} + \sigma_r \epsilon^r_{t-1}$. The estimated persistence of the quarterly real rate ($\rho_r$) is 0.95, which translates into $\alpha_r = (1 - \rho_r)/ (3 \times \Delta t) = 0.2$, where $\Delta t = 1/12$. The standard deviation ($\sigma_r$) is equal to 0.410.

### Dealers Cost of Capital: $\eta$

We approximate the cost of capital of dealers, $\eta$, as follows. For each one of the five largest US banks by Assets, we obtain the current (as of August 2018) credit rating from Fitch, Standard and Poor’s and Moody’s. At the same time we obtain AA, A, BBB daily option-adjusted spreads (OAS) for US corporate bonds from FRED Economic Data for the period January 1st of 1997 to August 27th of 2018. The source of the data is: [https://fred.stlouisfed.org/search?st=ICE+BofAML+US+Corporate+AA+Option-Adjusted+Spread](https://fred.stlouisfed.org/search?st=ICE+BofAML+US+Corporate+AA+Option-Adjusted+Spread). Options adjusted spreads measure the spread of a family of US Corporate issuers of the same credit rating adjusting for...
Table 2: Summary Baseline Calibration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>Sovereign’s risk aversion</td>
<td>2</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Coupon rate</td>
<td>0.04</td>
</tr>
<tr>
<td>$\alpha_y$</td>
<td>Persistence of output</td>
<td>0.2</td>
</tr>
<tr>
<td>$\alpha_r$</td>
<td>Persistence of short rate</td>
<td>0.2</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Arrival rate</td>
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</tr>
<tr>
<td>$\eta$</td>
<td>Cost of Capital for Intermediaries (pct/py)</td>
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</tr>
<tr>
<td>$\lambda$</td>
<td>Implied liquidity cost</td>
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</tr>
<tr>
<td>$\rho$</td>
<td>Discount Factor</td>
<td>0.0416</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Poisson Int. of a Large Shock (pct/py)</td>
<td>2.00</td>
</tr>
<tr>
<td>$\Delta y$</td>
<td>Drop in output (pct)</td>
<td>5.00</td>
</tr>
<tr>
<td>$\Delta r$</td>
<td>Increase in rates (pct)</td>
<td>1.00</td>
</tr>
<tr>
<td>$\varsigma$</td>
<td>Logistic p.d.f. scale parameter</td>
<td>100</td>
</tr>
</tbody>
</table>

any embedded option.  

We denote $OAS_t(Rating)$ as a function that maps a rating and a moment in time to the option-adjusted spread. For each bank $i$ we fix the current credit rating, $Rating_{i,t}$, for the entire sample of daily observations at the final value, $Rating_{i,T}$. Then, we obtain a time series for the spread of each bank as $\eta_{i,t} = OAS_t(Rating_{i,t})$ for $t = 1\ldots T$. We weight the spreads of each bank by their relative assets (as of 2018). Finally, we obtain an estimated value of the spread of intermediaries in our model as $\hat{\eta} = \frac{\sum_{i=1}^{5} \sum_{T=1}^{T} \omega_i \eta_i}{\sum_{i=1}^{5} \omega_i}$, equal to 149 basis points.

**Spanish Debt Profile: Figure 2.1**

For the maturity debt profile of Spain featured in figure 2.1, we use data from the Spanish Treasury. The data can be accessed at [www.tesoro.es/en/deuda-publica/estadisticas-mensuales](http://www.tesoro.es/en/deuda-publica/estadisticas-mensuales) and corresponds to the debt profile as of July the 31st of 2018. We use the total debt.

**Large Shock Intensity: $\phi$**

We calibrate the intensity of the shock based on the data from Barro and Ursua (2010). This data-set is obtained from [https://scholar.harvard.edu/barro/publications/barro-ursua-macroeconomic-data](https://scholar.harvard.edu/barro/publications/barro-ursua-macroeconomic-data). Out of 1600 year-country observations for OECD countries, 34 of them correspond to an output drop of more than 5 percent. This amounts to an estimated frequency equal to 2.13 percent per year; one large shock every 50 years. We calibrate the arrival of shocks in the risky steady state, $\phi$, to a value equal to 2.00 percent per year.

**Large Shock Size: $\Delta y(0), \Delta r(0)$**

We fix 5 percent as the size of the shock to output, and we denote it as $\Delta y(0)$. For interest rates, the shock to the short rate is the one that implies the same drop in consumption than with the shock to output, given the persistence of rates (as well as the other parameters). This is an increase in rates from 4 to 5 percent, and we denote it as $\Delta r(0)$.

---

Bank of America Merrill Lynch computes the index. More precisely, the option-adjusted spread for bonds is “the number of basis points that the fair value government spot curve is shifted to match the present value of discounted cash flows to the bond price. For securities with embedded options, such as call, sink or put features, a log-normal short interest rate model is used to evaluate the present value of the securities potential cash flows.” Thus, the index fits a term structure model for different securities, to separate the value of the security from any embedded option by subtracting or adding the value of the option.
Scale Logistic Distribution: \( \zeta \)

To calibrate \( \zeta \) we match the unconditional default probability of Spain during the period 1877-1982. We proceed as follows. According to Barro and Ursua (2010), for the period 1945 to 2009 the most significant year-to-year drop in income for Spain was 4.8 percent. Thus, we will fix the size of the shock output to 5.0 percent and the intensity to \( \phi = 0.02 \), as in section 3. The preference shocks are distributed according to a logistic distribution with a probability density function given by:

\[
f(\varepsilon) = \frac{\zeta e^{-\zeta \varepsilon}}{(1 + e^{-\zeta \varepsilon})^2}.
\]

We set the scale parameter, \( \zeta \), equal to 100. As we mentioned in the main text, for our calibration, this value of the parameter produces a default in 32 percent of the events when an extreme shock hits Spanish output. Given the intensity of the extreme shock, \( \phi = 0.02 \), this implies an unconditional default probability equal to 0.6 percent per year, roughly a default every 157 years. This is in line with the findings of Reinhart and Rogoff (2009) in which Spain experienced one default during the period 1877-1982.

E Some stylized facts bond issuances in Spain

We report data on the monthly gross issuances of bonds by the Spanish Government. The period covers from January 2000 to December 2018. The source of the data is the Spanish Treasury. The data can be accessed at [www.tesoro.es/en/deuda-publica/historico-de-estadisticas/subastas-2001-2014](http://www.tesoro.es/en/deuda-publica/historico-de-estadisticas/subastas-2001-2014). In this period the Treasury issued debt in bills (zero-coupon bonds) and regular bonds of different maturities. Bills are issued at 3, 6, 12 and 18-month maturities, and bonds are issued at 3, 5, 10, 15 and 30-year maturities. Not all maturities are active every month. For example, in the period 2014-2018 the 18-month bill and the 30-year bond were not issued. We construct a panel of annual issuances over GDP by accumulating individual gross issuances, and dividing them by the Spanish GDP in current euros. The source for GDP data is FRED economic data [https://fred.stlouisfed.org/series/CLVMNACSCAB1GQES](https://fred.stlouisfed.org/series/CLVMNACSCAB1GQES).

Figure E.1 reports the yearly average of issuances at each maturity over GDP. A pattern can be observed from the data. (i) Maturity is increasing for all bonds up to one year. (ii) For bonds from 3 years up to 10 years, maturity is also increasing, even if the absolute levels are below the amount of 1-year bonds. (iii) The issuance of 15 and 30-year bonds is roughly similar and relatively small. The figure also displays issuances weighted by the total issuance over GDP of the corresponding year. The pattern is similar independently of the weighting.

Another remarkable feature is that this pattern has been relatively stable over time. Figure E.2 presents the yearly averages for each vintage. Notwithstanding the changes in the total level of issuances, there is a remarkable stability in the issuance pattern that we highlight in the figure E.1. For instance, the bottom right panel displays the issuances in the period 2014-2018 of low interest rates and quantitative easing. Notice how, even if issuances have shifted towards long maturities, the issuance pattern remains roughly similar to the average.

With a liquidity coefficient \( \bar{\lambda} \) constant across maturities our model cannot replicate this pattern of maturity increasing within groups of bonds and discrete issuances. However, a natural way to fit the data, without over-parameterizing the model, would be to allow for discrete issuances, as outlined in section 5, and to consider heterogeneity in the liquidity coefficient among different groups of bonds. For example, we could let order flows (captured by \( \mu \)) to differ for bonds with maturities below 1 year, between 1 and 10 years, and above 10 years. This variation of the model can be rationalized if bonds trade in markets that differ in their liquidity or ability to be

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31The Treasury has occasionally issued at other maturities, including recent issuances of 50-year bonds, but we do not consider them in the analysis.
pledged as collateral (see Krishnamurthy and Vissing-Jorgensen, 2012), or if they are catered to customers with different investment horizons (see, for example, Vayanos and Vila, 2009).
Figure E.1: Issuance by maturity as a percentage GDP. Sample average for Spain 2000 - 2018
Figure E.2: Issuance by maturity as a percentage of GDP. Vintages 2000 - 2018
F Computational method

We describe the numerical algorithm used to jointly solve for the equilibrium domestic valuation, $v(\tau, t)$, bond price, $q(t, \tau, \iota)$, consumption $c(t)$, issuance $i(\tau, t)$ and density $f(\tau, t)$. The initial distribution is $f(\tau, 0) = f_0(\tau)$. The algorithm proceeds in 3 steps. We describe each step in turn.

**Step 1: Solution to the domestic value** The steady state equation (2.10) is solved using an upwind finite difference scheme similar to Achdou et al. (2017). We approximate the valuation $v_{ss}(\tau)$ on a finite grid with step $\Delta \tau \in \{\tau_1, ..., \tau_I\}$, where $\tau_i = \tau_{i-1} + \Delta \tau = \tau_1 + (i - 1) \Delta \tau$ for $2 \leq i \leq I$. The bounds are $\tau_1 = \Delta \tau$ and $\tau_I = T$, such that $\Delta \tau = T/I$. We use the notation $v_i := v_{ss}(\tau_i)$, and similarly for the issuance $i_i$. Notice first that the domestic valuation equation involves first derivatives of the valuations. At each point of the grid, the first derivative can be approximated with a forward or a backward approximation. In an upwind scheme, the choice of forward or backward derivative depends on the sign of the drift function for the state variable. As in our case, the drift is always negative, we employ a backward approximation in state:

$$\frac{\partial v(\tau_i)}{\partial \tau} \approx \frac{v_i - v_{i-1}}{\Delta \tau}, \quad (F.1)$$

The equation is approximated by the following upwind scheme,

$$\rho v_i = \delta + \frac{v_{i-1} - v_i}{\Delta \tau},$$

with terminal condition $v_0 = v(0) = 1$. This can be written in matrix notation as

$$\rho \mathbf{v} = \mathbf{u} + \mathbf{A} \mathbf{v},$$

where

$$\mathbf{A} = \frac{1}{\Delta \tau} \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{I-1} \\ v_I \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \delta - 1/\Delta \tau \\ \delta \\ \delta \\ \vdots \\ \delta \end{bmatrix}. \quad (F.2)$$

The solution is given by

$$\mathbf{v} = (\rho \mathbf{I} - \mathbf{A})^{-1} \mathbf{u}. \quad (F.3)$$

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as $\mathbf{A}$.

To analyze the transitional dynamics, define $t^{\text{max}}$ as the time interval considered, which should be large enough to ensure a converge to the stationary distribution and time is discretized as $t_n = t_{n-1} + \Delta t$, in intervals of length

$$\Delta t = \frac{t^{\text{max}}}{N - 1},$$

where $N$ is a constant. We use now the notation $v^n_i := v(\tau_i, t_n)$. The valuation at $t^{\text{max}}$ is the stationary solution computed in (F.3) that we denote as $\mathbf{v}^N$. We choose a forward approximation in time. The dynamic value equation (2.10) can thus be expressed

$$r^n \mathbf{v}^n = \mathbf{u} + \mathbf{A} \mathbf{v}^n + \frac{(\mathbf{v}^{n+1} - \mathbf{v}^n)}{\Delta t},$$

36
where $r^n := r(t_n)$. By defining $B^n = \left( \frac{1}{\Lambda} + r^n \right) I - A$ and $d^{n+1} = u + \frac{\psi^{n+1}}{\Lambda}$, we have

$$v^n = (B^n)^{-1} d^{n+1}, \quad (F.4)$$

which can be solved backwards from $n = N - 1$ until $n = 1$.

The optimal issuance is given by

$$\iota^n_i = \frac{1}{\lambda} \left( \frac{\psi^n_i - \bar{\psi}_i^n}{\psi^n_i} \right),$$

where $\psi_i^n$ is computed in an analogous form to $v_i^n$.

**Step 2: Solution to the Kolmogorov Forward equation** Analogously, the KFE equation (2.1) can be approximated as

$$\frac{f^n_i - f^{n-1}_i}{\Delta t} = \iota^n_i + \frac{f^n_{i+1} - f^n_i}{\Delta \tau},$$

where we have employed the notation $f^n_i := f(\tau_i, t_n)$. This can be written in matrix notation as:

$$\frac{f^n - f^{n-1}}{\Delta t} = \iota^n + A^T f^n, \quad (F.5)$$

where $A^T$ is the transpose of $A$ and

$$f^n = \begin{bmatrix} f^n_1 \\ f^n_2 \\ \vdots \\ f^n_{I-1} \\ f^n_I \end{bmatrix}, \quad \iota^n = \begin{bmatrix} \iota^n_1 \\ \iota^n_2 \\ \vdots \\ \iota^n_{I-1} \\ \iota^n_I \end{bmatrix}.$$ 

Given $f_0$, the discretized approximation to the initial distribution $f_0(\tau)$, we can solve the KF equation forward as

$$f_n = \left( I - \Delta t A^T \right)^{-1} (\iota^n \Delta t + f_{n-1}), \quad n = 1,.., N. \quad (F.6)$$

**Step 3: Computation of consumption** The discretized budget constraint (2.2) can be expressed as

$$c^n = \bar{y}^n - f^n_1 + \sum_{i=1}^{I} \left[ \left( 1 - \frac{1}{2} \lambda^n_i \right) \iota^n_i \psi^n_i - \delta f_i^n \right] \Delta \tau, \quad n = 1,.., N.$$

Compute

$$r^n = \rho + \sigma \frac{c^{n+1} - c^n}{\Delta t}, \quad n = 1,.., N - 1.$$

**Complete algorithm** The algorithm proceeds as follows. First guess an initial path for consumption, for example $c^n = \bar{y}^n$, for $n = 1,.., N$. Set $k = 1$;

**Step 1:** Issuances. Given $c_{k-1}$ solve step 1 and obtain $\iota$.

**Step 2:** KF. Given $\iota$ solve the KF equation with initial distribution $f_0$ and obtain the distribution $f$.

**Step 3:** Consumption. Given $\iota$ and $f$ compute consumption $c$. If $\|c - c_{k-1}\| = \sum_{n=1}^{N} |c^n - c^n_{k-1}| < \epsilon$ then stop. Otherwise compute

$$c_k = \omega c + (1 - \omega) c_{k-1}, \quad \lambda \in (0,1),$$

set $k := k + 1$ and return to step 1.
G Solutions without liquidity costs: risk and default

G.1 Risk without liquidity costs

We now consider the case without liquidity costs, \( \bar{\lambda} = 0 \). With positive liquidity costs, adjustments in portfolios are costly. By studying the problem at the limit where liquidity costs are zero, we can understand the extent to which the government can obtain insurance given the set of bonds it has available. Thus, it clarifies the extent to which liquidity costs limit insurance.

Toward that goal, we note that the necessary conditions of the problem are the same with and without liquidity costs, including the issuance rule. If the issuance rule holds, issuances are bounded if and only if valuations and prices are equal, \( v = \psi \) and \( \phi = \hat{\psi} \). If we substitute \( v = \psi \) and \( \phi = \hat{\psi} \) in the PDE for valuations, equation (3.3), and subtract the bond PDE from both sides, equation (3.1), we obtain a premium condition that must hold for all bonds:

\[
\hat{r} (t) - \phi \mathbb{E}_t^X \left[ \frac{U' (c (t))}{U' (\hat{c} (t))} \cdot \frac{\psi (\tau, t)}{\hat{\psi} (\tau, t)} \right] = \hat{r}^* (t) - \phi \mathbb{E}_t^X \left[ \frac{\psi (\tau, t)}{\hat{\psi} (\tau, t)} \right].
\]  

(G.1)

The analysis of the different solutions of equation (G.1) provides useful information about the role of hedging and self-insurance. We analyze each case in turn.

Perfect hedging: replicating the complete-markets allocation. Equation (G.1) replicates the complete-markets allocation when consumption follows a continuous path, i.e., when \( \hat{c} (t^p) = c (t^p; X (t^p)) \) for any realization of the shock. This is because international investors are risk-neutral and the government is risk averse. If consumption does not jump, condition (G.1) implies \( \hat{r} = \hat{r}^* \). In a complete markets economy consumption growth satisfies \( \frac{\hat{c} (t)}{\hat{c} (t)} = \frac{\hat{r} (t) - \rho}{\hat{\sigma}} \), the same rule that it follows in a deterministic problem. Naturally, there is no RSS with positive consumption if \( \hat{r}^* (t) < \rho \), but consumption converges asymptotically toward zero.

Consumption does not jump when it is possible to form a perfect hedge, a debt profile that generates a capital gain that exactly offsets the shock. Any shock changes the net-present value of income. Given the path of rates, the optimal consumption rule and the initial post-shock consumption produce a net-present value of consumption. A perfect hedge thus produces the capital gains such the net present value of consumption minus income at the time of the shock (denoted by \( \Delta B (t^p, X (t^p)) \)) is covered to the point where pre- and post-shock consumption are equal: \( c (t^p) = \hat{c} (t^p; X (t^p)) \). This must be true for any shock, \( X (t^p) \). In the context of the model, a perfect hedge exists if the debt distribution satisfies at all times \( t \)

\[
\Delta B (t, X (t)) = - \int_0^T (\psi (\tau, t; X (t)) - \hat{\psi} (\tau, t)) \hat{f} (\tau, t) d\tau,
\]  

(G.2)

for any possible realization of \( X (t) \). This family of equations is a generalization of the discrete-shock and discrete-bonds matrix conditions that guarantee market completion in Duffie and Huang (1985), Angeletos (2002) or Buera and Nicolini (2004).

In our model, perfect hedging is available in the case of an interest rate shock taking \( N \) possible values. Then, there is continuum of solutions that satisfy equation (G.2). In this case we can use a range of maturities \([0, T]\) that is as short as we want to hedge. The shorter the range, the more extreme the positions we obtain. A second observation has to do with the direction of hedges. Consider the case of a single jump in interest rates (\( N = 1 \)). To

\[32\]This equation is recovered also by solving the problem with \( \lambda = 0 \) directly. The proof is available upon request.

\[33\]In the case of discrete shocks and discrete bonds, the existence of complete-markets solution requires the presence of at least \( N + 1 \) bonds for \( N \) shocks. In the case of a continuum of shocks, the condition requires the invertibility of a linear operator. Proving conditions on \( G \) that guarantee that family of solutions exceeds the scope of the paper. In this case, equation (G.2) is just a system of \( N \) linear integral equations, for every \( t \), known as Fredholm equations of the first kind.
offset the reduction in the net-present value of income, the debt profile must generate an increase in wealth. This requires an increase in short-term assets and long-term liabilities.

No hedging: only self insurance. The opposite to the complete-markets outcome is the case of income shocks. In this case bond prices do not change, \( \psi = \hat{\psi} = 1 \). Therefore, it is not possible to generate capital gains with a debt profile. Instead, the only solution to (G.1) is:

\[
\frac{\dot{c}(t)}{c(t)} = \frac{\hat{\rho}(t) + \phi \left( E^X \left[ \frac{U'\left(c(t)\right)}{U\left(c(t)\right)} \right] - 1 \right)}{\sigma}.
\]

This is a situation in which no hedging is available, because the asset space does not allow any form of external insurance. Instead, the government must self-insure. Self insurance is captured by the ratio of marginal utilities which effectively lowers \( \hat{\rho}(t) \). To solve for consumption, this extreme case coincides with a single-bond economy without interest-rate risk. The jump in consumption is given by the jump in the net present value of income. The solution to \( c(t) \) in this case is known and can be found, for example in Wang et al. (2016). The ratio of marginal utilities in the solution increases as the level of assets falls. This means that, provided there is a sufficiently low level of debt, the economy reaches a RSS with positive consumption. The convergence in consumption is a manifestation of self-insurance.

General case. The general case with both income and interest rates shocks described by equation (G.1) features an intermediate point between the two extreme cases described above as both a partial hedging and self-insurance emerge.\(^{34} \) Furthermore, as long as the support of the shocks has cardinality \( N \) the debt profile is indeterminate, as only \( N \) points of the debt distribution are pinned down.\(^{35} \)

G.2 Default without liquidity costs

We return to the case without liquidity costs, \( \lambda = 0 \), but allow for default. Without liquidity costs, we have again that a solution necessarily features equality between valuations and prices, \( v = \psi \) and \( \hat{v} = \hat{\psi} \). As a result, the condition that characterizes the solution without default, (G.1), is modified to:

\[
\hat{\rho}(t) - \phi E^X \left[ \Theta(V(t)) + \Omega(t) \right] \frac{\psi(\tau, t)}{\psi(\tau, t)} \left( \frac{U'\left(c(t)\right)}{U\left(c(t)\right)} \right) = \hat{\rho}(t) - \phi E^X \left[ \Theta(V(t)) \frac{\psi(\tau, t)}{\psi(\tau, t)} \right]. \quad (G.3)
\]

As in the case without default, we can explain how condition (G.3) characterizes the solution depending on the set of bonds and shocks.

On the impossibility of perfect hedging. The presence of default interrupts the ability to share risk. Efficient risk sharing requires a continuous consumption path along non-default states. To see how default interrupts risk-sharing, consider the case where interest-rate shocks allow complete asset spanning. Assume that \( \dot{c}(t) = c(t) \) holds in non-default states, as in the version without default. In this case, condition (G.3) becomes

\[
\dot{\hat{\rho}}(t) - \phi E^X \left[ \Omega(t) \frac{\psi(\tau, t)}{\psi(\tau, t)} \right] = \hat{\rho}(t).
\]

This equation is not satisfied if two maturities feature a different price jump. However, full asset spanning requires a different price jump at two maturities. This contradiction implies that even when the set of securities can provide

\(^{34}\) This case can be solved via dynamic programming using aggregate debt at market values \( \hat{B} \) as a state variable. \( \hat{B} \) is defined as in (2.17) with pre-shock prices and debt profile. Equation (G.1) holds for every maturity, so given \( \hat{B} \), it represents a family of first-order conditions for \( f(\tau, t) \). The debt profile then is associated with an insurance cost of \( \hat{B} \).

\(^{35}\) The proof is a particular case of the one presented in Appendix C.8 for the case with default.
insurance in non-default states, the government’s solution with commitment does not adopt a perfectly insuring scheme. The distortion follows because the echo-effect acts differently than the risk premium, it distorts valuations but not prices—we can see that even under risk-neutrality.

**Default allows some hedging.** Consider the case of only income shocks. Without default, we noted that there was no hedging role for maturity but now we show that with default, there is a role. Default opens the possibility of a partial hedging because prior to the shock, different maturities are priced differently. Post-shock prices are always $\psi(\tau, t^o) = 1$. This means that once a shock hits, the government can exploit the change in the yield curve to obtain capital gains in its portfolio. The change in the risk premium is akin to the spanning effect of an interest-rate jump.

**General case.** The option to default interrupts insurance across non-default states, but allows price variation even without interest-rate risk. As long as the cardinality of shocks is discrete, the maturity profile is indeterminate—a formal proof is found in Appendix C.8. One extreme case of indeterminacy is that of a shock which does not produce a jump in income nor interests, but only grants a default option. Aguiar et al. (Forthcoming) studies that shock in a discrete-time model similar to ours but without commitment.

### H Additional figures

In this section we introduce the figures corresponding to the different exercises performed in the case of a $T \to \infty$, risk-neutral governments or turning off the revenue-echo effect ($\Omega = 0$).
Figure H.1: Asymptotic equilibrium objects as a function of the maximum maturity $(T)$. 
Figure H.2: Response to an unexpected shock to interest rates with $\sigma = 0$. 
Figure H.3: Response to a shock to interest rates with $\sigma = 0$. ‘Baseline’ stands for the model starting at the baseline RSS and $'\sigma = 0'$ for the case with a risk-neutral government.
Figure H.4: Response to a shock to income with $\sigma = 0$ when the option to default is available. Panels (e) and (f) refer to the case with $\sigma = 0$. 

Panels (a) and (b) show the debt distribution $f(\tau)$ and consumption $c(t)$, respectively, with and without the option to default. Panels (c) and (d) illustrate the total debt $b(t)$ and average duration, while panels (e) and (f) depict bond prices $\psi(\tau,t)$ and domestic valuations $v(\tau,t)$. 

The graphs demonstrate how different parameters affect the economic outcomes, with a focus on the impact of the default option on the distribution of debt and consumption patterns.