A Proofs

A.1 Proof of Proposition 1

A.1.1 The planning problem

Existence of a solution. Consider the space of square integrable measurable trades (γ, φ). Social welfare can be written

$$W(\gamma, \varphi \,|\, N) = \mathbb{E}[v] \int \left[\omega(x) + \varphi(x) + \int \gamma(x, x') dN(x' \,|\, \text{otc})\right] dN(x) - \frac{\eta}{2} \mathbb{V}[v] \int \left[\omega(x) + \varphi(x) + \int \gamma(x, x')\, dN(x' \,|\, \text{otc})\right]^2 dN(x).$$

Given that the measure N is finite, $N(X) < \infty$, repeat applications of the Cauchy Schwartz inequality show that the social welfare function is continuous in (γ, φ).

Because $(\gamma, \varphi) \mapsto W(\gamma, \varphi \,|\, N)$ is continuous, it is lower semi-continuous. Clearly, the function is also concave and the constraint set is bounded. Existence of a solution then follows from an application of Proposition 1.2, Chapter II in Eckland and Témam (1987).

Almost everywhere uniqueness of $g(x)$. Because the objective is strictly concave, all solutions must share the same $g(x)$, almost everywhere according to N.

Post-trade exposures are constant almost everywhere over $X_{\textbf{cent}}$. Otherwise, given strict concavity, one could achieve a strictly higher welfare by pooling the exposures of all centralized market participants. Given that $M(x, x')$ is bounded on the support of N, and given that pre-trade exposures are bounded, the centralized trades required to pool exposures are also bounded, hence feasible.

First-order conditions. Take any optimal γ and let

$$\begin{aligned}\hat{\gamma}(x, x') &= \gamma(x, x') + \varepsilon \left[M(x, x') - \gamma(x, x')\right] \mathbb{I}_{\{g(x) < g(x')\}} - \varepsilon \left[M(x', x) + \gamma(x, x')\right] \mathbb{I}_{\{g(x) > g(x')\}} \\ &\equiv \gamma(x, x') + \varepsilon \Delta(x, x').\end{aligned}$$

One easily sees that $\hat{\gamma}$ is feasible for the planning problem, as long as $\varepsilon \in [0, 1]$. Hence, for small ε, we obtain that up to second-order terms:

$$
\begin{aligned}
&\frac{W(\hat{\gamma},\varphi)-W(\gamma,\varphi)}{N(X_{\text{otc}})} = \varepsilon \int\int U'\left[g(x)\right]\Delta(x,x')\,dN(x\,|\,\text{otc})\,dN(x'\,|\,\text{otc})\\
=&\frac{\varepsilon}{2}\int\int U'\left[g(x)\right]\Delta(x,x')\,dN(x\,|\,\text{otc})\,dN(x'\,|\,\text{otc})\\
&+\frac{\varepsilon}{2}\int\int U'\left[g(x')\right]\Delta(x',x)\,dN(x'\,|\,\text{otc})\,dN(x\,|\,\text{otc})\\
=&\frac{\varepsilon}{2}\int\int \left\{U'\left[g(x)\right]-U'\left[g(x')\right]\right\}\Delta(x,x')\,dN(x\,|\,\text{otc})\,dN(x'\,|\,\text{otc})\\
=&\frac{\varepsilon}{2}\int\int \left\{U'\left[g(x)\right]-U'\left[g(x')\right]\right\}\left[M(x,x')-\gamma(x,x')\right]\mathbb{I}_{\{g(x)<g(x')\}}\,dN(x\,|\,\text{otc})\,dN(x'\,|\,\text{otc})\\
&-\frac{\varepsilon}{2}\int\int \left\{U'\left[g(x)\right]-U'\left[g(x')\right]\right\}\left[M(x',x)+\gamma(x,x')\right]\mathbb{I}_{\{g(x)>g(x')\}}\,dN(x\,|\,\text{otc})\,dN(x'\,|\,\text{otc}).
\end{aligned}
$$

If $\gamma(x,x')$ is optimal, this must be negative. Since both integrands are positive, they must be zero almost everywhere, or:

$$
N(\cdot\,|\,\text{otc}) \text{ a.e. over } X^2_{\text{otc}},\ \gamma(x,x') = \begin{cases} M(x,x') & \text{if } g(x)<g(x') \\ \in[-M(x',x),M(x,x') & \text{if } g(x)=g(x') \\ -M(x',x) & \text{if } g(x)>g(x') \end{cases}
$$

A.1.2 Some preliminary results

In this section we study what can be viewed as the "partial equilibrium" determination of post-trade exposures $g(x)$. That is, we consider the problem of determining the post-trade exposure of an individual bank, taking as given any arbitrary distribution of post-trade exposures in the OTC market. This preliminary result is important because it allows us to determine the post-trade exposures of any (ω,k), even those who choose not to participate in the OTC market.

Formally, we consider some arbitrary $x\in X$, and we fix some arbitrary function for the post-trade exposures, $h(x')$, of other banks in the OTC market. We then seek a solution to the problem:

$$
g = \omega(x) + \int_{X_{\text{otc}}} \gamma(x,x')\,dN(x'\,|\,\text{otc}), \tag{19}
$$

where, for all $x'\in X_{\text{otc}}$:

$$
\gamma(x,x') = \begin{cases} M(x,x') & \text{if } g<h(x') \\ \in[-M(x',x),M(x,x')] & \text{if } g=h(x') \\ -M(x',x) & \text{if } g>h(x') \end{cases} \tag{20}
$$

To that end we define the following two functions. First:

$$\overline{V}(g) = \omega(x) + \int M(x,x')\mathbb{I}_{\{g\le h(x')\}}\, dN(x' \,|\, \text{otc}) - \int M(x',x)\mathbb{I}_{\{g>h(x')\}}\, dN(x' \,|\, \text{otc})$$
$$= \omega(x) + \int \left[M(x,x') + M(x',x)\right] \mathbb{I}_{\{g\le h(x')\}}\, dN(x' \,|\, \text{otc}) - \int M(x',x)\, dN(x' \,|\, \text{otc}).$$

The function $\overline{V}(g)$ represents the maximum post-trade exposure of the bank of type x, if all its traders take position anticipating that the post-trade will be g. One easily sees that $\overline{V}(g)$ is decreasing and left-continuous. Second, we let:

$$\underline{V}(g) = \omega(x) + \int M(x,x')\mathbb{I}_{\{g<h(x')\}}\, dN(x' \,|\, \text{otc}) - \int M(x',x)\mathbb{I}_{\{g\ge h(x')\}}\, dN(x' \,|\, \text{otc})$$
$$= \omega(x) + \int M(x',x)\, dN(x' \,|\, \text{otc}) - \int \left[M(x,x') + M(x',x)\right] \mathbb{I}_{\{g\ge h(x')\}}\, dN(x' \,|\, \text{otc}).$$

The function $\underline{V}(g)$ represents the minimum post-trade exposure of a bank active in the OTC market only, if all its traders take position anticipating that the post-trade will be g. One easily sees that $\underline{V}(g)$ is decreasing and right-continuous. One also sees easily that:

$$\overline{V}(g^+) = \underline{V}(g) \text{ and } \overline{V}(g) = \underline{V}(g^-)$$

where the notation g^+ and g^- is for right- and left-limit. Finally, given that pre-trade exposures, $\omega(x)$, are bounded, and given that $M(x,x')$ is bounded over the support of N, there exists $a < b$ such that $\underline{V}(g) \in [a,b]$ and $\overline{V}(g) \in [a,b]$ for all g.

We then have:

Lemma 6. *A post-trade exposure g solves (19)-(20) if and only if $g \in [\underline{V}(g), \overline{V}(g)]$.*

Proof. For the "only if" part, take a solution of (19)-(20) and use the optimality conditions (20) to show that it belongs to $[\underline{V}(g), \overline{V}(g)]$. For the "if" part, take some $g(x) \in [\underline{V}(g), \overline{V}(g)]$, let $\gamma(x,x') = M(x,x')$ if $g(x) < h(x')$, let $\gamma(x,x') = -M(x',x)$ if $g(x) > h(x')$, and let $\gamma(x,x') = \alpha M(x,x') - (1-\alpha)M(x',x)$ if $g = h(x')$, where α is chosen so that

$$g = \omega(x) + \int M(x,x')\mathbb{I}_{\{g<h(x')\}}\, dN(x' \,|\, \text{otc}) - \int M(x',x)\mathbb{I}_{\{g>h(x')\}}\, dN(x' \,|\, \text{otc})$$
$$+ \int \left[\alpha M(x,x') - (1-\alpha)M(x',x)\right] \mathbb{I}_{\{g=h(x')\}}\, dN(x' \,|\, \text{otc}).$$

Given that $g \in [\underline{V}(g), \overline{V}(g)]$, it follows that $\alpha \in [0,1]$, hence $\gamma(x,x') \in [-M(x',x), M(x,x')]$ if $g = h(x')$. □

Next we show that:

Lemma 7. *The fixed point problem $g \in [\underline{V}(g), \overline{V}(g)]$ has a unique solution.*

Proof. To show that a solution exists, we apply Kakutani's Fixed Point Theorem (see, e.g., Theorem 7 in Nachbar, 2017) to the correspondence

$$g \rightrightarrows V(g) \equiv [\underline{V}(g), \overline{V}(g)].$$

It is clear that $V(g)$ takes values that are convex sets included in $[a, b]$. To see that $V(g)$ has a closed graph consider any converging sequence $(g_n, v_n) \to (g, v)$ such that $v_n \in V(g_n)$ for all n. Then we can extract a subsequence of g_n such that either $g_n \leq g$ for all n or $g_n \geq g$ for all n. Suppose that we are in the former case (the latter case is symmetric). Then, since $\underline{V}(g)$ is decreasing, it follows that $v_n \geq \underline{V}(g_n) \geq \underline{V}(g)$. Going to the limit, we obtain $v \geq \underline{V}(g)$. Since $v_n \leq \overline{V}(g_n)$ and since $\overline{V}(g)$ is left-continuous, it follows that $v \leq \overline{V}(g)$. Therefore, $v \in V(g)$.

For uniqueness consider any $g \in V(g)$. Then $g \geq \underline{V}(g) = \overline{V}(g^+)$. But since $\overline{V}(g)$ is decreasing, it follows that $g' > \overline{V}(g')$ for all $g' > g$. Hence, $g' \notin V(g')$. A similar argument applies to $g' < g$. □

Finally, we obtain

Lemma 8. *The solution of the fixed point problem* (19)-(20) *remains the same:*

- *If the post-trade exposure function h is changed but remains the same $N(\,\cdot\,|\,\text{otc})$-almost everywhere;*
- *If the optimality conditions are required to hold $N(\,\cdot\,|\,\text{otc})$-almost everywhere.*

This follows directly because these change do not impact the functions $\underline{V}(g)$ and $\overline{V}(g)$, hence the fixed point remains the same.

A.1.3 Equilibrium existence

We construct an equilibrium from a solution of the planning problem. The main difficulty in doing so is that the planner's problem determines trades only N-almost everywhere, that is, only for investors' types who are actually present in the market. An equilibrium, by contrast, requires that trade be well defined for all types.

Consider, then a solution (γ, φ) to the planning problem, and the associated post-trade exposures g. These may not constitute an equilibrium because the equilibrium conditions only hold almost everywhere. To obtain an equilibrium based on (γ, φ, g), we modify these functions on a set on measure zero so that the equilibrium conditions do hold everywhere. The key part of the modification is to determine the post-trade exposure $g(x)$ for measure zero subsets of X_{otc}, i.e. for the (ω, k) who choose not to participate in the OTC market. For this we rely on the result of the previous subsection: given the post-trade exposures of types who do participate, the post-trade exposure of any other $x \in X_{\text{otc}}$ is uniquely determined.

Step 1: modify trades of $x \in X_{\text{cent}}$. From the planner's problem we know that there exists some constant g_{cent} such that $g(x) = g_{\text{cent}}$, N-almost everywhere for $x \in X_{\text{cent}}$. For any $x \in X_{\text{cent}}$ such that $g(x) \neq g_{\text{cent}}$ we pick $\gamma(x, x') = 0$ if $(x, x') \notin X^2_{\text{otc}}$, and otherwise we pick $\gamma(x, x')$ that satisfies (20) for all $x' \in X_{\text{otc}}$, given $g(x) = g_{\text{cent}}$. Finally, we let $\varphi(x) = g_{\text{cent}} - \omega(x) - \int \gamma(x, x') \, dN(x' \,|\, \text{otc})$. Because these changes are made for a measure zero set of x, they do not impact the post-trade exposures of any other x', nor do they impact the market-clearing condition in the centralized market.

Step 2: modify trades of $(x, x') \in X^2_{\text{otc}}$. We define the following sets:

$$\begin{aligned}
\Phi &= \left\{(x, x') \in X^2_{\text{otc}} \text{ s.t. (20) holds for } (x, x')\right\} \\
\Phi(x) &= \left\{x' \in X_{\text{otc}} \text{ s.t. (20) holds for } (x, x')\right\} \\
\Psi &= \left\{x \in X_{\text{otc}} \text{ s.t. } N(\Phi(x) \,|\, \text{otc}) = 1\right\}.
\end{aligned}$$

One easily shows that $N(\Psi \,|\, \text{otc}) = 1$. Then, we define:

$$A = \Phi \cap (\Psi \times \Psi)$$

The set A has measure one because it is the intersection of two sets of measure one. It contains pairs (x, x') with the following properties. First, they together satisfy the optimality condition (20). Second, they each satisfy the optimality condition (20) with almost every other $\hat{x}$. Next we define:

$$\begin{aligned}
B &= \left\{x \in X_{\text{otc}} \,:\, (x, x') \in A \text{ for some } x'\right\} \\
C &= X_{\text{otc}} \backslash B.
\end{aligned}$$

The set B has also measure one because $A \subseteq B \times B$. Notice that any $x \in B$ is such that $N(\Phi(x) \,|\, \text{otc}) = 1$.

Our modification goes as follows:

- For all $(x, x') \in A$, the optimality condition (20) holds and so we keep $\gamma(x, x')$ the same.
- For all $(x, x') \in B^2$ but not in A, we modify $\gamma(x, x')$ so that it satisfies (20). Notice that since $N(\Phi(x) \,|\, \text{otc}) = N(\Phi(x') \,|\, \text{otc}) = 1$, these modifications concern a measure zero sets of counterparties for both x and x', and so they do not change the post-trade exposures $g(x)$ or $g(x')$.
- For all (x, x') such that $x \in C$ and $x' \in B$, we pick $\gamma(x, x')$ and $g(x)$ that solves the fixed point problem of Section A.1.2. For any $x' \in B$, this changes the bilateral trades for a measure zero set of counterparties and so does not change $g(x')$.

- For $(x, x') \in C^2$, then we change the bilateral trades so that they satisfy optimality condition (20). For either x or x', this changes the bilateral trades for a measure zero of counterparties, and so does not change $g(x)$ or $g(x')$.

A.2 Proof of Proposition 2

Given any admissible direction of reallocation, (n^+, n^-), let us define $n \equiv= n^+ - n^-$. Our maintained assumption ensure that $N + \varepsilon n \geq 0$ for all sufficiently small ε. For this proof we let $n \equiv= n^+ - n^-$ and we assume that:

$$N(X_{\text{otc}}) + \varepsilon\, n(X_{\text{otc}}) > 0 \tag{21}$$

$$N(X_{\text{cent}}) + \varepsilon\, n(X_{\text{cent}}) > 0, \tag{22}$$

for all sufficiently small $\varepsilon > 0$. These two conditions hold in particular under the maintained assumption of the proposition, namely, if there is strictly positive participation in the market and if (n^+, n^-) is an admissible direction of reallocation. However they are more general: they allow us also calculate the marginal social value of creating some positive participation in an empty market: e.g. $N(X_{\text{cent}}) = 0$ and $n(X_{\text{cent}}) > 0$.

Given $N + \varepsilon n$, the planner chooses a measurable function $\gamma : X^2 \to \mathbb{R}$ for bilateral exposures in the constraint set Γ such that:

$$\begin{aligned}
&\gamma(x, x') = 0 \text{ if } x \notin X_{\text{otc}} \\
&\gamma(x, x') + \gamma(x', x) = 0 \\
&- M(x', x) \leq \gamma(x, x') \leq M(x, x'),
\end{aligned}$$

where the function M is uniformly bounded in $(x, x') \in X^2$.

Given any γ and any ε, we define the post-otc exposure by:

$$h(x, \gamma, \varepsilon) = \omega(x) + \int \gamma(x, x') \, \frac{dN(x') + \varepsilon dn(x')}{N(X_{\text{otc}}) + \varepsilon n(X_{\text{otc}})}.$$

Then, the centralized market trade can be defined to "pool exposures," something we already know is optimal:

$$\varphi(x, \gamma, \varepsilon) = \int_{x' \in X_{\text{cent}}} \left[h(x', \gamma, \varepsilon) - h(x, \gamma, \varepsilon)\right] \frac{dN(x') + \varepsilon dn(x')}{N(X_{\text{cent}}) + \varepsilon n(X_{\text{cent}})}$$

if $x \in X_{\text{cent}}$, and $\varphi(x, \gamma, \varepsilon) = 0$ otherwise. Finally, we define the post-trade exposure by:

$$g(x, \gamma, \varepsilon) = h(x, \gamma, \varepsilon) + \varphi(x, \gamma, \varepsilon).$$

Social welfare before participation costs is:

$$W(\gamma,\varepsilon) = \int U\left[g(x,\gamma,\varepsilon)\right]\,\left(dN(x) + \varepsilon\,dn(x)\right).$$

Where the function W implicitly depends on $N + \varepsilon\,n$. The planner's problem can be defined in notations that closely follow Milgrom and Segal (2002):

$$W^\star(\varepsilon) = \sup_{\gamma\in\Gamma} W(\gamma,\varepsilon)$$
$$\Gamma^\star(\varepsilon) = \{\gamma\in\Gamma\,:\,W(\gamma,\varepsilon) = W^\star(\varepsilon)\}.$$

We know from our earlier results that the planner's problem has at least a solution, i.e. that $\Gamma^\star(\varepsilon)$ is not empty. The rest of the proof extends arguments from Milgrom and Segal (2002), and is organized as follows:

- In Section A.2.1 we show that the planner's value, $W^\star(\varepsilon)$, is right-hand differentiable at $\varepsilon = 0$.
- In Section A.2.2, we show that the right-hand derivative maximizes marginal social value with respect to all $\gamma \in \Gamma^\star$.

show that $W^\star$ is right-hand differentiable at ε, and that the right-hand derivative

A.2.1 Right-hand differentiability

To show that social welfare is right-hand differentiable, we check that the assumptions of Theorem 1 and 3 in Milgrom and Segal (2002) are satisfied in our setting. The main technical difficulty lies in establishing that the right-derivative can be calculated by taking the maximum of the partial derivative over all maximizers. This is a result that Milgrom and Segal (2002) provide in their Corollary 4 for "continuous functions on compact choice sets". Since our choice set is only weakly compact, we must check that the required continuity properties hold in the weak topology.

A useful preliminary result is that:

Lemma 9. *The functions h, φ, g, $\partial h/\partial\varepsilon$, $\partial\varphi/\varepsilon$, $\partial g/\partial\varepsilon$, $\partial^2 h/\partial\varepsilon^2$, $\partial^2\varphi/\varepsilon^2$, $\partial^2 g/\partial\varepsilon^2$, are all uniformly bounded in $(x,\gamma,\varepsilon)\in X\times\Gamma\times[0,\bar\varepsilon]$, for some $\bar\varepsilon$ small enough.*

Proof. For h, g and φ, this follows directly because γ and ω are uniformly bounded. The first derivatives of h, φ, g, can be calculated explicitly as:

$$\begin{aligned}\frac{\partial h}{\partial\varepsilon} &= \int \gamma(x,x')\,\frac{dn(x')N(X_{\text{otc}}) - dN(x')n(X_{\text{otc}})}{\left[N(X_{\text{otc}}) + \varepsilon\,n(X_{\text{otc}})\right]^2} \qquad (23)\\ \frac{\partial\varphi}{\partial\varepsilon} &= \mathbb{I}_{\{x\in X_{\text{cent}}\}}\Bigg\{\int\left[\frac{\partial h}{\partial\varepsilon}(x',\gamma,\varepsilon) - \frac{\partial h}{\partial\varepsilon}(x,\gamma,\varepsilon)\right]\frac{dN(x') + \varepsilon dn(x')}{N(X_{\text{cent}}) + \varepsilon n(X_{\text{cent}})}\\ &\quad + \int\left[h(x',\gamma,\varepsilon) - h(x,\gamma,\varepsilon)\right]\frac{dn(x')N(X_{\text{cent}}) - dN(x')n(X_{\text{cent}})}{\left[N(X_{\text{cent}}) + \varepsilon\,n(X_{\text{cent}})\right]^2}\Bigg\},\end{aligned}$$

and $\partial g/\partial\varepsilon = \partial h/\partial\varepsilon + \partial\varphi/\partial\varepsilon$. The result then follows because γ is uniformly bounded, because N and n are finite signed measures, and from (21) and (22), which ensure that the numerators, $N(X_{\text{otc}})+\varepsilon n(X_{\text{otc}})$ and $N(X_{\text{cent}})+\varepsilon n(X_{\text{cent}})$ are bounded away from zero for small enough ε. Similar arguments imply uniform boundedness of the second derivatives

$$\begin{aligned}
\frac{\partial^2 h}{\partial\varepsilon^2} =& -2n(X_{\text{otc}})\int \gamma(x,x')\,\frac{dn(x')N(X_{\text{otc}})-dN(x')n(X_{\text{otc}})}{\left[N(X_{\text{otc}})+\varepsilon\, n(X_{\text{otc}})\right]^3}\\
\frac{\partial^2\varphi}{\partial\varepsilon^2} =&\mathbb{I}_{\{x\in X_{\text{cent}}}\Bigg\{\int\left[\frac{\partial^2 h}{\partial\varepsilon^2}(x',\gamma,\varepsilon)-\frac{\partial h}{\partial\varepsilon}(x,\gamma,\varepsilon)\right]\frac{dN(x')+\varepsilon dn(x')}{N(X_{\text{cent}})+\varepsilon n(X_{\text{cent}})}\\
&2\int\left[\frac{\partial h}{\partial\varepsilon}(x',\gamma,\varepsilon)-\frac{\partial h}{\partial\varepsilon}(x,\gamma,\varepsilon)\right]\frac{dn(x')N(X_{\text{cent}})-dN(x')n(X_{\text{cent}})}{\left[N(X_{\text{cent}})+\varepsilon\, n(X_{\text{cent}})\right]^2}\Bigg\}\\
&-2n(X_{\text{cent}})\int\left[h(x',\gamma,\varepsilon)-h(x,\gamma,\varepsilon)\right]\frac{dn(x')N(X_{\text{cent}})-dN(x')n(X_{\text{cent}})}{\left[N(X_{\text{cent}})+\varepsilon\, n(X_{\text{cent}})\right]^3}\Bigg\},
\end{aligned}$$

and $\partial^2 g/\partial\varepsilon^2 = \partial^2 h/\partial\varepsilon^2 + \partial^2\varphi/\partial\varepsilon^2$. □

Uniform boundedness allows us to apply Leibniz' rule to differentiate under the integral sign, and obtain the first and second partial derivative of W with respect to ε:

$$\begin{aligned}
\frac{\partial W}{\partial\varepsilon} =& \int U\left[g(x,\gamma,\varepsilon)\right]\,dn(x)+\int\frac{dU}{dg}\left[g(x,\gamma,\varepsilon)\right]\frac{\partial g}{\partial\varepsilon}(x,\gamma,\varepsilon)\,\left[dN(x)+\varepsilon dn(x)\right]\\
\frac{\partial^2 W}{\partial\varepsilon^2} =&2\int\frac{dU}{dg}\left[g(x,\gamma,\varepsilon)\right]\frac{\partial g}{\partial\varepsilon}(x,\gamma,\varepsilon)\,dn(x)+\int\frac{d^2U}{dg^2}\left[g(x,\gamma,\varepsilon)\right]\left[\frac{\partial g}{\partial\varepsilon}(x,\gamma,\varepsilon)\right]^2\left[dN(x)+\varepsilon dn(x)\right]\\
&+\int\frac{dU}{dg}(x,\gamma,\varepsilon)\frac{\partial^2 g}{\partial\varepsilon^2}(x,\gamma,\varepsilon)\left[dN(x)+\varepsilon dn(x)\right].
\end{aligned}$$

The uniform boundedness properties of Lemma 9 then implies that both $\partial W/\partial\varepsilon$ and $\partial^2 W/\partial\varepsilon^2$ are uniformly bounded in (γ,ε). This further implies that both W and $\partial W/\partial\varepsilon$ are Lipchitz continuous functions of ε, with Lipchitz coefficients that do not depend on γ. Therefore, the equi-continuity and equi-differentiability properties required in Theorem 1 and 3 in Milgrom and Segal (2002) hold. It then follows from these two Theorems that:

Lemma 10. *Given any selection $\gamma^\star(\varepsilon)$ of the maximum correspondence:*

$$\frac{dW^\star}{d\varepsilon}(0^+) = \lim_{\varepsilon\to 0^+}\frac{\partial W}{\partial\varepsilon}(\gamma^\star(\varepsilon),\varepsilon) \geq \max_{\gamma^\star\in\Gamma^\star(0)}\frac{\partial W}{\partial\varepsilon}(\gamma^\star,0).$$

A.2.2 The right-hand derivative maximizes marginal social value

Next, we show the equality by adapting the argument of Corollary 4 in Milgrom and Segal. To that end consider a sequence $\varepsilon_m\to 0^+$ and some associated sequence of bilateral exposures $\gamma^\star_m\in\Gamma^\star(\varepsilon_m)$. Let $h^\star_m(x)\equiv h(x,\gamma^\star_m,\varepsilon_m)$, $\varphi^\star_m(x)\equiv\varphi(x,\gamma^\star_m,\varepsilon_m)$, $g^\star_m(x)\equiv h^\star_m(x)+\varphi^\star_m(x)$, and $\partial g^\star_m/\partial\varepsilon(x)\equiv\partial g/\partial\varepsilon(x,\gamma^\star_m,\varepsilon_m)$. Similarly, let $h^\star(x)\equiv h(x,\gamma^\star,0)$, $\varphi^\star(x)\equiv\varphi(x,\gamma^\star,0)$, $g^\star(x)\equiv g(x,\gamma^\star,0)$ and

$\partial g^\star/\partial\varepsilon = \partial g/\partial\varepsilon(x,\gamma^\star,0)$.

Weak convergence. Given that that bilateral exposures are uniformly bounded, the Riez Weak Compactness Theorem (Royden and Fitzpatrick, 2010, Section 19.4) allows us to successively extract weakly convergent subsequences, so that we can assume without loss of generality that γ_m converges weakly to some $\gamma^\star$ in $L^2(N\times N)$, $L^2(N\times n)$, $L^2(n\times N)$ and $L^2(n\times n)$, and that the sequences of real numbers $\int U\left[g^\star_m(x)\right]\,dN(x)$, $\int U\left[g^\star_m(x)\right]\,dn(x)$ and $\int dU/dg[g^\star_m(x)]\partial g^\star_m/\partial\varepsilon(x)\,dN(x)$ all converge. It then follows from direct calculations using the explicit formula for partial derivatives shown in the proof of Lemma 9 that $h^\star_m(x)$, $\varphi^\star_m(x)$, $g^\star_m(x)$, $\partial g^\star_m/\partial\varepsilon(x)$ converge to $h^\star(x)$, $\varphi^\star(x)$, $g^\star(x)$, $\partial g^\star/\partial\varepsilon(x)$ weakly in $L^2(N)$ and $L^2(n)$.

Strong convergence and asymptotic optimality of post-trade exposures. Given that $g\mapsto\int U\left[g(x)\right]\,dN(x)$ is strongly continuous and convex, it is weakly upper semi-continuous (see Corollary 2.2 in Eckland and Témam, 1987), which implies that:

$$\int U\left[g^\star(x)\right]\,dN(x) \geq \lim_{m\to\infty}\int U\left[g^\star_m(x)\right]\,dN(x). \tag{24}$$

Given any $\gamma\in\Gamma$, the optimality of $\gamma^\star_m$ given the distribution $N+\varepsilon_m n$ implies that:

$$\int U\left[g^\star_m(x)\right]\,\left[dN(x)+\varepsilon_m dn(x)\right] \geq \int U\left[g(x,\gamma,\varepsilon_m)\right]\,\left[dN(x)+\varepsilon_m dn(x)\right]. \tag{25}$$

It can be easily checked that, holding γ fixed, $g(x,\gamma,\varepsilon_m)\to g(x,\gamma,0)$ strongly in $L^2(N)$. Given that $g\mapsto\int U\left[g(x)\right]\,dN(x)$ is strongly continuous, we can go to the limit in the inequality (25) and, combining with (24), we obtain:

$$\int U\left[g^\star(x)\right]\,dN(x) \geq \lim_{m\to\infty}\int U\left[g^\star_m(x)\right]\,dN(x) \geq \int U\left[g(x,\gamma,0)\right]\,dN(x).$$

It follows that $\gamma^\star$ is an optimum for $\varepsilon=0$, i.e. $\gamma^\star\in\Gamma^\star(0)$. Taking the supremum over $\gamma\in\Gamma$ implies that $\lim_{m\to\infty}U\left[g^\star_m(x)\right]\,dN(x)=\int U\left[g^\star(x)\right]\,dN(x)$. Since $U\left[g\right]$ is quadratic and $g^\star_m\to g^\star$ weakly in $L^2(N)$, it follows that $\int\left[g^\star_m(x)\right]^2\,dN(x)\to\int\left[g^\star(x)\right]^2\,dN(x)$. Therefore $g^\star_m\to g^\star$ weakly in $L^2(N)$, and the $L^2(N)$ norm of g_m converges to that of $g^\star$. It thus follows that $g^\star_m\to g^\star$ strongly in $L^2(N)$.

The derivative maximizes marginal social value. With these results in mind, consider

$$\frac{\partial W}{\partial\varepsilon}(\gamma^\star_m,\varepsilon_m) = \int U\left[g^\star_m(x)\right]\,dn(x) + \int\frac{dU}{dg}\left[g^\star_m(x)\right]\frac{\partial g^\star_m}{\partial\varepsilon}\left[dN(x)+\varepsilon_m dn(x)\right]. \tag{26}$$

Using the weak upper semi continuity of $g \mapsto \int U\left[g(x)\right]\, dn(x)$ as above, we obtain that

$$\int U\left[g^\star(x)\right]\, dn(x) \geq \lim_{m\to\infty} U\left[g^\star_m(x)\right]\, dn(x). \tag{27}$$

Now recall that $dU/dg[g(x)]$ is linear, that $g^\star_m \to g^\star$ strongly in $L^2(N)$, that $\partial g^\star_m/\partial\varepsilon$ is uniformly bounded and converges weakly in $L^2(N)$ toward $\partial g^\star/\partial\varepsilon$. It thus follows that $dU/dg\left[g^\star_m\right]\partial g^\star_m/\partial\varepsilon$ converges weakly in $L^2(N)$ towards $dU/dg\left[g^\star\right]\partial g^\star/\partial\varepsilon$. Together with (27), this allows us to go to the limit as in (26) and obtain:

$$\frac{\partial W}{\partial\varepsilon}(\gamma^\star,0) = \int U\left[g^\star(x)\right]\, dn(x) + \int \frac{dU}{dg}\left[g^\star(x)\right]\frac{\partial g^\star}{\partial\varepsilon} dN(x) \geq \lim_{m\to\infty}\frac{\partial W}{\partial\varepsilon}(\gamma^\star_m,\varepsilon_m).$$

Combining with Lemma 10 we obtain

Lemma 11. *The right-hand derivative maximizes marginal social value:*

$$\frac{dW^\star}{d\varepsilon}(0^+) = \max_{\gamma^\star\in\Gamma^\star(0)}\frac{\partial W}{\partial\varepsilon}(\gamma^\star,0).$$

A.2.3 An expression of the partial derivative

Lemma 12. *For any* $\gamma\in\Gamma$*:*

$$\begin{aligned}\frac{\partial W}{\partial\varepsilon}(\gamma,0) = &\int U\left[g(x,\gamma,0)\right]\, dn(x)\\ &-\int\frac{dU}{dg}\left[\int_{X_{\text{cent}}} g(x',\gamma,0)d\nu(x')\right]\varphi(x,\gamma,0)\, dn(x)\\ &-\int\frac{dU}{dg}\left[g(x',\gamma,0)\right]\gamma(x,x')d\mu(x')\, dn(x)\\ &-\frac{n(X_{\text{otc}})}{2}\int\int\left\{\frac{dU}{dg}\left[g(x',\gamma,0)\right]-\frac{dU}{dg}\left[g(x'',\gamma,0)\right]\right\}\gamma(x',x'')d\mu(x')d\mu(x''),\end{aligned}$$

where

$$\begin{aligned}d\nu &= \frac{dN}{N(X_{\text{cent}})} \text{ if } N(X_{\text{cent}})>0 \text{ and } d\nu = \frac{dn}{n(X_{\text{cent}})} \text{ otherwise}\\ d\mu &= \frac{dN}{N(X_{\text{otc}})} \text{ if } N(X_{\text{otc}})>0 \text{ and } d\mu = \frac{dn}{n(X_{\text{otc}})} \text{ otherwise.}\end{aligned}$$

Notice that, in the formula, the measure used to calculate average depend on whether there is, under N, positive participation in a market. For instance, if $N(X_{\text{otc}}) > 0$, then the average is calculated based on the conditional distribution of incumbent in the OTC market. If $N(X_{\text{otc}}) = 0$, then the average is calculated based on the conditional distribution of entrant.

Proof. We first calculate:

$$\frac{\partial W}{\partial \varepsilon}(x,\gamma,0) = \int U\left[g(x)\right]\, dn(x) + \int \frac{dU}{dg}\left[g(x)\right] \frac{\partial g}{\partial \varepsilon}(x)\, dN(x)$$
$$= \int U\left[g(x)\right]\, dn(x) + \int \frac{dU}{dg}\left[g(x)\right] \frac{\partial h}{\partial \varepsilon}(x)\, dN(x) + \int \frac{dU}{dg}\left[g(x)\right] \frac{\partial \varphi}{\partial \varepsilon}(x)\, dN(x)$$

where, to simplify notations, we have let $g(x) \equiv g(x,\gamma,0)$ and $\partial g/\partial\varepsilon(x) \equiv \partial g/\partial \varepsilon\,(x,\gamma,0)$, and where we used that $g(x) = h(x) + \varphi(x)$. To help with the calculations, define:

$$A \equiv \int U\left[g(x)\right]\, dn(x) \tag{28}$$
$$B \equiv \int \frac{dU}{dg}\left[g(x)\right] \frac{\partial h}{\partial \varepsilon}(x)\, dN(x)$$
$$C \equiv \int \frac{dU}{dg}\left[g(x)\right] \frac{\partial \varphi}{\partial \varepsilon}(x)\, dN(x).$$

An expression for B**.** We first work on:

$$B = \int \frac{dU}{dg}\left[g(x)\right] \frac{\partial h}{\partial \varepsilon}(x)\, dN(x).$$

If $N(X_{\text{otc}}) = 0$, then one sees from its definition that $h(x,\gamma,\varepsilon)$ does not depend on ε, hence $\partial h/\partial\varepsilon = 0$ and $B = 0$ as well. If $N(X_{\text{otc}}) > 0$, then the formula (23) implies that

$$\int\int \frac{dU}{dg}\left[g(x)\right] \frac{\partial h}{\partial \varepsilon}(x)\, dN(x)$$
$$= \int\int \frac{dU}{dg}\left[g(x)\right] \gamma(x,x') \frac{dn(x')N(X_{\text{otc}}) - dN(x')n(X_{\text{otc}})}{N(X_{\text{otc}})^2}\, dN(x)$$
$$= \int\int \frac{dU}{dg}\left[g(x')\right] \gamma(x',x) \frac{dN(x')}{N(X_{\text{otc}})}\, dn(x)$$
$$- n(X_{\text{otc}}) \int\int \frac{dU}{dg}\left[g(x)\right] \gamma(x,x') \frac{dN(x)\, dN(x')}{N(X_{\text{otc}})^2}$$
$$= -\int\int \frac{dU}{dg}\left[g(x')\right] \gamma(x,x') \frac{dN(x')}{N(X_{\text{otc}})}\, dn(x)$$
$$- n(X_{\text{otc}}) \int\int \frac{dU}{dg}\left[g(x)\right] \gamma(x,x') \frac{dN(x)\, dN(x')}{N(X_{\text{otc}})^2} \tag{29}$$

where the second-to-last equality follows by exchanging the name of variables, namely replacing x by x' in the first term, and the last equality follows by bilateral feasibility, i.e. $\gamma(x',x) = -\gamma(x,x')$. The second term can be simplified as follows:

$$
\begin{aligned}
&- n(X_{\text{otc}}) \int\int \frac{dU}{dg}\,[g(x)]\,\gamma(x,x')\frac{dN(x)\,dN(x')}{N(X_{\text{otc}})^2} \\
= &- \frac{n(X_{\text{otc}})}{2}\int\int \frac{dU}{dg}\,[g(x)]\,\gamma(x,x')\frac{dN(x)\,dN(x')}{N(X_{\text{otc}})^2} \\
&- \frac{n(X_{\text{otc}})}{2}\int\int \frac{dU}{dg}\,\left[g(x')\right]\gamma(x',x)\frac{dN(x)\,dN(x')}{N(X_{\text{otc}})^2} \\
= &- \frac{n(X_{\text{otc}})}{2}\int\int \frac{dU}{dg}\,[g(x)]\,\gamma(x,x')\frac{dN(x)\,dN(x')}{N(X_{\text{otc}})^2} + \\
&\frac{n(X_{\text{otc}})}{2}\int\int \frac{dU}{dg}\,\left[g(x')\right]\gamma(x,x')\frac{dN(x)\,dN(x')}{N(X_{\text{otc}})^2} \\
= &- \frac{n(X_{\text{otc}})}{2}\left\{\frac{dU}{dg}\,[g(x)] - \frac{dU}{dg}\,\left[g(x')\right]\right\}\gamma(x,x')\frac{dN(x)\,dN(x')}{N(X_{\text{otc}})^2},
\end{aligned}
$$

where: the first equality follows by breaking the integral into two identical halves and exchanging the name of variables in the second term, replacing x by x' in the second half; the second equality follows by bilateral feasibility $\gamma(x',x) = -\gamma(x,x')$; and the third equality by collecting terms. Taken together we obtain that, if $N(X_{\text{otc}}) > 0$:

$$
\begin{aligned}
B = &- \int\int \frac{dU}{dg}\,\left[g(x')\right]\gamma(x,x')d\mu(x)\,dn(x) \\
&- \frac{n(X_{\text{otc}})}{2}\int\int\left\{\frac{dU}{dg}\,\left[g(x')\right] - \frac{dU}{dg}\,\left[g(x'')\right]\right\}\gamma(x',x'')d\mu(x')d\mu(x''),
\end{aligned}
\tag{30}
$$

where $d\mu(x) = dN(x)/N(X_{\text{otc}})$.

While formula (30) has been derived assuming $N(X_{\text{otc}}) > 0$, one easily check that it is also valid for $N(X_{\text{otc}}) = 0$. Indeed, using equation (29) and setting $d\mu(x) = dn(x)/n(X_{\text{otc}})$, one obtains that (30) is equal to zero, which is the value of B when $N(X_{\text{otc}}) = 0$.

An expression for C. Next, we turn to

$$
C = \int \frac{dU}{dg}\,[g(x)]\,\frac{\partial\varphi}{\partial\varepsilon}(x)\,dN(x).
$$

If $N(X_{\text{cent}}) = 0$, then clearly $C = 0$, since $\varphi(x) = 0$ if $x \notin X_{\text{cent}}$. If $N(X_{\text{cent}}) > 0$, then $g(x)$ constant over $x \in X_{\text{cent}}$ and evidently equal to

$$
\bar{g} = \int_{X_{\text{cent}}} g(x')\frac{dN(x')}{N(X_{\text{cent}})},
$$

for all $x \in X_{\text{cent}}$. Hence

$$\begin{aligned}
C =& \frac{dU}{dg}(\bar{g}) \int \frac{\partial \varphi}{\partial \varepsilon}(x)\, dN(x) \\
=& \frac{dU}{dg}(\bar{g}) \int\int_{X^2_{\text{cent}}} \left[\frac{\partial h}{\partial \varepsilon}(x') - \frac{\partial h}{\partial \varepsilon}(x)\right] \frac{dN(x')\, dN(x)}{N(X_{\text{cent}})} \\
&+ \frac{dU}{dg}(\bar{g}) \int\int_{X^2_{\text{cent}}} \left[h(x') - h(x)\right] \frac{dn(x')N(X_{\text{cent}}) - dN(x')n(X_{\text{cent}})}{N(X_{\text{cent}})^2}\, dN(x).
\end{aligned}$$

Note that, on a symmetric domain such as X^2_{cent}, integrals of the form $\int\int [f(x') - f(x)]\, dN(x')dN(x)$ are equal to zero. Hence we are only left with:

$$\begin{aligned}
C =& \frac{dU}{dg}(\bar{g}) \int\int_{X^2_{\text{cent}}} \left[h(x') - h(x)\right] dn(x') \frac{dN(x)}{N(X_{\text{cent}})} \\
=& -\frac{dU}{dg}(\bar{g}) \int\int_{(X^2_{\text{cent}}} \left[h(x') - h(x)\right] \frac{dN(x')}{N(X_{\text{cent}})}\, dn(x) \\
=& -\frac{dU}{dg}(\bar{g}) \int \varphi(x)\, dn(x) \qquad (31)
\end{aligned}$$

where the second equality follows from exchanging the name of variables, replacing x by x', and the third equality follows by definition of $\varphi(x)$.

While formula (31) has been derived assuming $N(X_{\text{cent}}) > 0$, one easily check that it is also valid for $N(X_{\text{cent}}) = 0$. Indeed, in this case, one easily check based on the formula for $\varphi(x, \gamma, \varepsilon)$ that $\int \varphi(x)\, dn(x) = 0$. Hence, formula (31) is valid both for $N(X_{\text{cent}}) > 0$ and $N(X_{\text{cent}}) = 0$.

Collecting terms. Adding up A, B and C given in equation (28), (30) and (31), we arrive at the formula of the lemma.

□

A.2.4 Equilibrium exposures maximize the partial derivative

To proceed we assume that $N(X_{\text{otc}}) > 0$ and $N(X_{\text{cent}}) > 0$.[27]

We first show that equilibrium bilateral exposures maximize $\frac{\partial W}{\partial \varepsilon}(\gamma, 0)$ with respect to $\gamma \in \Gamma^\star(0)$. Recall first the expression for the partial derivative we derived in the previous section:

[27] While the proposition restricts attention to the case of strictly positive participation in both market, it can be extended to the case of $N(X_{\text{otc}}) = 0$ or $N(X_{\text{cent}}) = 0$, after redefining the equilibrium in an appropriate way. Consider for example participation patterns such that $N(X_{\text{otc}}) = 0$ and $N(X_{\text{cent}}) > 0$, with a perturbation such that $n(X_{\text{otc}}) > 0$. Then, one needs to define equilibrium post-trade exposures when there is a large group of investor, of size $N(X_{\text{cent}} \setminus X_{\text{otc}})$, participating in the centralized market, and a "infinitesimal" group of investors, with type distribution $n(x)/n(X_{\text{otc}})$, participating in the OTC and possibly simultaneously in the centralized market.

$$\frac{\partial W}{\partial \varepsilon}(\gamma, 0) = \int U\left[g(x)\right]\, dn(x) - \int \frac{dU}{dg}\left[\int_{X_{\text{cent}}} g(x')\frac{dN(x')}{N(X_{\text{cent}})}\right]\varphi(x,\gamma,0)\, dn(x) \tag{32}$$
$$- \int \frac{dU}{dg}\left[g(x')\right]\gamma(x,x')\frac{dN(x')}{N(X_{\text{otc}})}\, dn(x)$$
$$- \frac{n(X_{\text{otc}})}{2}\int\int\left\{\frac{dU}{dg}\left[g(x')\right] - \frac{dU}{dg}\left[g(x'')\right]\right\}\gamma(x',x'')\frac{dN(x')dN(x'')}{N(X_{\text{otc}})^2},$$

where, as before, we let $g(x) = g(x,\gamma,0)$ to simplify notations. Now consider any socially optimal $\gamma \in \Gamma^\star(0)$. Because the first-order conditions hold almost everywhere according to $N(\,\cdot\,|\,X_{\text{otc}}) \times N(\,\cdot\,|\,X_{\text{otc}})$, it follows that the integrand of the last term is equal to

$$\left\{\frac{dU}{dg}\left[g(x')\right] - \frac{dU}{dg}\left[g(x'')\right]\right\}^+ M(x',x'') + \left\{-\frac{dU}{dg}\left[g(x')\right] + \frac{dU}{dg}\left[g(x'')\right]\right\}^+ M(x'',x'),$$

almost everywhere according to $N(\,\cdot\,|\,X_{\text{otc}}) \times N(\,\cdot\,|\,X_{\text{otc}})$. Therefore, the last term is constant and equal to

$$-n(X_{\text{otc}})\frac{\bar{F}}{2},$$

for any socially optimal $\gamma \in \Gamma^\star(0)$. We also recall that socially optimal post-trade exposures, $g(x)$ are uniquely determined almost everywhere according to N, which implies that:

$$\bar{g} \equiv \int_{X_{\text{cent}}} g(x')\frac{dN(x')}{N(X_{\text{cent}})}$$

is also a constant.

Now let γ denote an equilibrium bilateral exposures, and let $\hat{\gamma}$ denote any socially optimal bilateral exposures. We calculate:

$$U\left[g(x)\right] - \frac{dU}{dg}\left[\bar{g}\right]\varphi(x) - \int \frac{dU}{dg}\left[g(x')\right]\gamma(x,x')\frac{dN(x')}{N(X_{\text{otc}})}$$
$$- U\left[\hat{g}(x)\right] + \frac{dU}{dg}\left[\bar{g}\right]\hat{\varphi}(x) + \int \frac{dU}{dg}\left[\hat{g}(x')\right]\hat{\gamma}(x,x')\frac{dN(x')}{N(X_{\text{otc}})}$$
$$\geq \frac{dU}{dg}\left[g(x)\right]\left[g(x) - \hat{g}(x)\right] - \frac{dU}{dg}\left[\bar{g}\right]\varphi(x) + \frac{dU}{dg}\left[\bar{g}\right]\hat{\varphi}(x)$$
$$- \int \frac{dU}{dg}\left[g(x')\right]\left\{\gamma(x,x') - \hat{\gamma}(x,x')\right\}\frac{dN(x')}{N(X_{\text{otc}})}$$
$$\geq \left\{\frac{dU}{dg}\left[g(x)\right] - \frac{dU}{dg}\left[\bar{g}\right]\right\}\left[\varphi(x) - \hat{\varphi}(x)\right]$$
$$+ \int \left\{\frac{dU}{dg}\left[g(x)\right] - \frac{dU}{dg}\left[g(x')\right]\right\}\left\{\gamma(x,x') - \hat{\gamma}(x,x')\right\}\frac{dN(x')}{N(X_{\text{otc}})},$$

where: the first inequality follows by concavity, and because $g(x') = \hat{g}(x')$ almost everywhere according to N; the second inequality follows using $g(x) = h(x) + \varphi(x)$ as well as the explicit expression of $h(x)$ in terms of $\gamma(x, x')$.

The first term on the right-side of the last inequality is zero. Indeed if $x \notin X_{\text{cent}}$, then $\hat{\varphi}(x) = \varphi(x) = 0$. If $x \in X_{\text{cent}}$ then, by construction, $g(x) = \bar{g}$. The second term on the right-side of the last inequality is positive because in equilibrium, the first-order conditions hold everywhere. Hence we have shown that the integrand of (32) is greatest when evaluated at equilibrium bilateral exposures, and the result follows.

A.2.5 The partial derivative in terms of marginal social value

Our calculations so far show that the partial derivative evaluated at equilibrium bilateral exposures is equal to:

$$
\begin{aligned}
\frac{\partial W}{\partial \varepsilon}(\gamma, 0) &= \int U\left[g(x)\right]\, dn(x) - \int \frac{dU}{dg}\left[\bar{g}\right] \varphi(x)\, dn(x) \\
&\quad - \int \frac{dU}{dg}\left[g(x')\right] \gamma(x, x') \frac{dN(x)}{N(X_{\text{otc}})}\, dn(x) - n(X_{\text{otc}}) \frac{\bar{F}}{2} \\
&= \int U\left[g(x)\right]\, dn(x) - \int \frac{dU}{dg}\left[g(x)\right] \varphi(x)\, dn(x) \\
&\quad - \int \frac{dU}{dg}\left[g(x')\right] \gamma(x, x') \frac{dN(x)}{N(X_{\text{otc}})}\, dn(x) - \int \mathbb{I}_{\{x \in X_{\text{otc}}\}} \frac{\bar{F}}{2}\, dn(x),
\end{aligned}
$$

where the second equality follows because either $\varphi(x) = 0$, or $\varphi(x) \neq 0$ and $g(x) = \bar{g}$, which implies that $\frac{dU}{dg}\left[\bar{g}\right] \varphi(x) = \frac{dU}{dg}\left[g(x)\right] \varphi(x)$. Next we add and subtract:

$$
\begin{aligned}
&\int U\left[\omega(x)\right]\, dn(x) + \int \frac{dU}{dg}\left[g(x)\right] \left\{h(x) - \omega(x)\right\}\, dn(x) \\
=& \int U\left[\omega(x)\right]\, dn(x) + \int\int \frac{dU}{dg}\left[g(x)\right] \gamma(x, x') \frac{dN(x')}{N(X_{\text{otc}})}\, dn(x)
\end{aligned}
$$

Collecting terms we obtain:

$$
\begin{aligned}
\frac{\partial W}{\partial \varepsilon}(\gamma, 0) &= \int U\left[\omega(x)\right]\, dn(x) \\
&\quad + \int \left\{ U\left[g(x)\right] - U\left[\omega(x)\right] - \frac{dU}{dg}\left[g(x)\right] \left\{g(x) - \omega(x)\right\} \right\}\, dn(x) \\
&\quad + \int\int \left\{ \frac{dU}{dg}\left[g(x)\right] - \frac{dU}{dg}\left[g(x')\right] \right\} \gamma(x, x') \frac{dN(x')}{N(X_{\text{otc}})}\, dn(x) \\
&\quad - \int \mathbb{I}_{\{x \in X_{\text{otc}}\}} \frac{\bar{F}}{2}\, dn(x),
\end{aligned}
$$

and the result follows by observing that equilibrium trades satisfy the first-order conditions everywhere and by re-defining function $\varepsilon \mapsto W^\star\left[N+\varepsilon\left(n^+-n^-\right)\right]$ to capture the participation costs as well:

$$W^\star\left[N+\varepsilon\left(n^+-n^-\right)\right](\varepsilon)=\sup_{\gamma\in\Gamma}\int\left\{U\left[g(x,\gamma,\varepsilon)\right]-C\left[\pi\left(x\right)\right]\right\}\left[dN(x)+\varepsilon\left(dn^+(x)-dn^-(x)\right)\right].$$

A.3 Proof of Lemma 2

The lemma starts with the intuitive result that banks in the centralized market have the same post-trade exposure. Given symmetry in participation patterns, this common post-trade exposure has to be 1/2.

The atom property. Let us guess and verify that post-trade exposures conditional on participating in the OTC market have the "atom property": all banks with the same endowment ω who participate in the OTC markets have the same post-trade exposures, regardless of their capacity. Formally, for each $\omega\in\{0,1\}$, $g(\omega,k,\text{otc})$ is independent of $k\in[0,1]$.[28] In addition, we conjecture that risk-sharing is imperfect, that is $\omega=0$-banks do not equalize their post-trade exposures with $\omega=1$-banks, $g(0,k,\text{otc})<1/2<g(1,k,\text{otc})$.

If the atom property holds, and if there is imperfect risk sharing, the optimality condition (9) implies that, when an $\omega=0$ trader is paired with an $\omega=1$ trader, the $\omega=0$ trader buys $\max\{k,k'\}$ units from $\omega=1$ trader. When two $\omega=0$-traders meet, they anticipate the same post-trade exposures and so are indifferent between any quantity in $[-\max\{k,k'\},\max\{k,k'\}]$. The post-trade exposure of an $\omega=0$ bank conditional on participating in the OTC market can thus be written:

$$g(0,k,\text{otc})=\frac{1}{2}\int\gamma(k,k')\,dN(k'\,|\,X_{\text{otc}})+\frac{1}{2}\int\max\left\{k,k'\right\}dN(k'\mid X_{\text{otc}}),\tag{33}$$

where, with a slight abuse of notation, "$\gamma(k,k')$" stands for "$\gamma\left[(0,k,\text{otc}),(0,k',\text{otc})\right]$." The first term is the net trade with $\omega=0$ banks. The second term is the net trade with $\omega=1$ banks. Now if we aggregate (33) over $k\in[k^\star,1]$, the bilateral feasibility constraint (2) implies that the aggregate net trade between $\omega=0$ banks is equal to zero. Therefore, if the atom property holds, we must have:

$$g(0,k,\text{otc})=\int g(0,k',\text{otc})\,dN(k'\,|\,X_{\text{otc}})=\frac{1}{2}\mathbb{E}\left[\max\{k',k''\}\,|\,(k',k'')\in X^2_{\text{otc}}\right],\tag{34}$$

where, with a slight abuse of notation, "$k'\in X_{\text{otc}}$" stands for "$(0,k',\text{otc})\in X_{\text{otc}}$". In words, the post-trade exposure of $\omega=0$ bank is equal to $\frac{1}{2}$, which is the probability of a meeting between an

[28]Notice that this property is assumed to hold for all k. That is, if an ω-bank makes the possibly suboptimal choice to participate in the OTC market, it will equalize its exposure with that other ω banks in the OTC market.

$\omega = 0$ and an $\omega = 1$ traders, multiplied by $\mathbb{E}\left[\max\{k', k''\} \,|\, (k', k'') \in X_{\text{otc}}^2\right]$, which is the average trade size between an $\omega = 0$ and an $\omega = 1$ trader.

To verify our guess, we need to find bilateral trades between $\omega = 0$-traders, $\gamma(k, k')$, that have two properties. First, they must satisfy the bilateral trading capacity constraint (4) for all (k, k'). Second, the post-trade exposures resulting from these bilateral trades, (34), must be equalized. A natural candidate for these bilateral trades is:

$$\gamma(k, k') = \mathbb{E}\left[\max\{k', k''\} \,|\, k'' \in X_{\text{otc}}\right] - \mathbb{E}\left[\max\{k, k''\} \,|\, k'' \in X_{\text{otc}}\right].$$

That is, when two $\omega = 0$-traders meet, they "swap" the exposures their banks acquired from $\omega = 1$ banks. When aggregated across all possible $\omega = 0$ counterparties, these swaps mechanically equalize exposure of all $\omega = 0$ banks who participate in the OTC market. To see that these swaps also satisfy the bilateral trading capacity constraint, note that

$$\max\{k', k''\} - \max\{k, k''\} = \max\{k' - k'', 0\} - \max\{k - k'', 0\} \in [-k, k'],$$

and so satisfies the bilateral trading capacity constraint.

Similarly, one can show that the atom property holds for $\omega = 1$-banks as well with $g(1, k, \text{otc}) = 1 - g(0, k, \text{otc})$. We have thus verified our conjecture that the post-trade exposures of OTC market participants are independent of k. That small-k (resp. large-k) banks trade like customers (resp. dealers) is proved in the main text.

A.4 Proof of Proposition 3

First, for equation (16) to have a solution, we need that the left-hand side is larger than the right-hand side at $k^\star = 0$, and smaller at $k^\star = 1$, which can be written $0 \leq C(\text{cent}) - C(\text{otc}) \leq \frac{|U_{gg}|}{36}$.

Now given a solution $k^\star$ to the (16), Lemma 3 implies that banks with $k \leq k^\star$ prefer the centralized market to the OTC market, while banks with $k \geq k^\star$ prefer the OTC market to the centralized market. To ensure that they prefer participating in some market to stay in autarky, we need that:

$$\max\{\text{MPV}(\omega, k, \text{cent}) - C(\text{cent}), \text{MPV}(\omega, k, \text{otc}) - C(\text{otc})\} \geq 0$$

for all $k \in [0, 1]$. Lemma 3 implies that this function is increasing in k, hence it is positive if and only if it is positive at $k = 0$. But when $k = 0$, the bank prefers the centralized market to the OTC market. We thus conclude that banks prefer to participate in some market than to stay in autarky if and only if $C(\text{cent}) \leq \frac{|U_{gg}|}{8}$. Finally, we can rule out participation in two market by setting $C(\text{otc+cent})$ large enough, and we are done.

A.5 Proof of Lemma 5

Direct calculations show that:

$$\frac{d}{d\omega}K(\omega,k,\text{otc}) = |U_{gg}|\,(g-\omega)\left(\frac{dg}{d\omega}-1\right)$$
$$\frac{d}{d\omega}F(\omega,k,\text{otc}) = |U_{gg}|k\left[2N(\{\omega'<\omega\}\,|\,X_{\text{otc}})-1\right]\frac{dg}{d\omega},$$

where g and $dg/d\omega$ denote, respectively, the post-trade exposure and its right-derivative for an ω-bank who participates in the OTC market. Using that $g-\omega = k\,[1-2N(\{\omega'<\omega\}\,|\,X_{\text{otc}})]$, we obtain:

$$\frac{d}{d\omega}\text{MPV}(\omega,k,\text{otc}) = -|U_{gg}|k\left[1-2N(\{\omega'<\omega\}\,|\,X_{\text{otc}})\right]\left[\frac{1}{2}\frac{dg}{d\omega}-1\right].$$

Now use that $\frac{d}{d\omega}\text{MPV}(\omega,k,\text{cent}) = -|U_{gg}|\left(\frac{1}{2}-\omega\right)$, and obtain:

$$\frac{d}{d\omega}\left[\text{MPV}(\omega,k,\text{cent})-\text{MPV}(\omega,k,\text{otc})\right]$$
$$= -|U_{gg}|\left\{\frac{1}{2}-\omega+k\left[1-2N(\{\omega'<\omega\}\,|\,X_{\text{otc}})\right]\left[\frac{1}{2}\frac{dg}{d\omega}-1\right]\right\}$$

For $\omega<\omega^\star$, then $dg/d\omega=1$, $N(\{\omega'<\omega\}\,|\,X_{\text{otc}})=0$, so that:

$$\frac{d}{d\omega}\left[\text{MPV}(\omega,k,\text{cent})-\text{MPV}(\omega,k,\text{otc})\right] = -|U_{gg}|\left\{\frac{1}{2}-\omega-\frac{k}{2}\right\}<0,$$

given that $\omega<\omega^\star$ and $\omega^\star+k<\frac{1}{2}$. Next, for $\omega\in\left[\omega^\star,\frac{1}{2}\right]$, $N(\{\omega'<\omega\}\,|\,X_{\text{otc}}) = \frac{\omega-\omega^\star}{1-2\omega^\star}$ and $\frac{dg}{d\omega} = 1-\frac{2k}{1-2\omega^\star}$. Substituting and rearranging, we obtain after a few lines of algebra:

$$\frac{d}{d\omega}\left[\text{MPV}(\omega,k,\text{cent})-\text{MPV}(\omega,k,\text{otc})\right] = -|U_{gg}|\frac{1-2\omega}{1-2\omega^\star}\left(\frac{1}{2}-\omega^\star-k\right)\left(\frac{1}{2}+\frac{k}{1-2\omega^\star}\right)\leq 0,$$

with equality if $\omega=\frac{1}{2}$.

Using the symmetry, i.e.,

$$MPV(1-\omega,k,\text{cent}) = MPV(\omega,k,\text{cent})$$
$$MPV(1-\omega,k,\text{otc}) = MPV(\omega,k,\text{otc}),$$

one easily sees that, for $\omega\in\left[\frac{1}{2},1\right]$

$$\frac{d}{d\omega}\left[\text{MPV}(\omega,k,\text{cent})-\text{MPV}(\omega,k,\text{otc})\right]\geq 0,$$

with equality if $\omega=\frac{1}{2}$.

A.6 Proof of Proposition 4

Two conditions. The left side of equation (18) is zero at $\omega = \frac{1}{2} - k$ and it is strictly decreasing for $\omega \in \left[0, \frac{1}{2} - k\right]$. Hence, a necessary and sufficient condition for (18) to have a solution in $\left[0, \frac{1}{2} - k\right]$ is that the left-side evaluated at $\omega = 0$ is greater than the right side:

$$\frac{|U_{gg}|}{8}(1 - 2k) > C(\text{cent}) - C(\text{otc}). \tag{35}$$

Once $\omega^\star$ is found, Lemma 5 ensures that banks in $\omega \in [0, \omega^\star]$ prefer the centralized market to the OTC market, and banks in $\omega \in \left[\omega^\star, \frac{1}{2}\right]$ prefer the OTC market to the centralized market (incentives are symmetric for $\omega \geq \frac{1}{2}$).

We next need to make sure that banks prefer their choices of market to autarky.

$$\max\left\{\text{MPV}(\omega, k, \text{cent}) - C(\text{cent}), \text{MPV}(\omega, k, \text{otc}) - C(\text{otc})\right\} > 0.$$

Clearly, this function is decreasing since both MPV's are decreasing. Therefore, the condition holds if and only if it holds for $\omega = \frac{1}{2}$.

$$\begin{aligned}
\text{MPV}\left(\frac{1}{2}, k, \text{otc}\right) - C(\text{otc}) > 0 &\Leftrightarrow \frac{1}{2}F\left(\frac{1}{2}, k, \text{otc}\right) - C(\text{otc}) > 0 \\
&\Leftrightarrow \frac{|U_{gg}|k}{4}\left(\frac{1}{2} - \omega^\star - k\right) - C(\text{otc}) > 0.
\end{aligned} \tag{36}$$

The first equivalence follows because, for an $\omega = \frac{1}{2}$-bank, the competitive surplus is zero. The second equivalence because the frictional surplus is proportional to the average distance between the post-trade exposure of the $\omega = \frac{1}{2}$ bank, and the post-trade exposures of other banks. Given uniform distribution and symmetry, this average distance is equal to half the distance between the post trade exposure of the $\omega = \frac{1}{2}$ bank, $g\left(\frac{1}{2}, k, \text{otc}\right) = \frac{1}{2}$, and the post-trade exposure of the $\omega^\star$-bank, $g(\omega^\star, k, \text{otc}) = \omega^\star + k$.

Conditions (35) and (36) in the C(otc), C(cent) plane. Condition (35) can be re-written:

$$C(\text{cent}) < C(\text{otc}) + \frac{|U_{gg}|}{8}(1 - 2k),$$

i.e., $C(\text{cent})$ must be below a line with slope one and intercept, $\frac{|U_{gg}|}{8}(1 - 2k)$. Condition (36) requires some work because it depends on $\omega^\star$, which is itself a function of $C(\text{otc})$ and $C(\text{cent})$. To obtain a workable representation, note first that (36) can be equivalently written as

$$\omega^\star \leq \frac{1}{2} - k - \frac{4C(\text{otc})}{|U_{gg}|k},$$

where $\omega^\star$ solves (18). Since the left-hand side of (18) is strictly decreasing in $\omega < \frac{1}{2} - k$, this is equivalent to requiring that, when evaluated at $\frac{1}{2} - k - \frac{4C(\text{otc})}{|U_{gg}|k}$ the left-hand side of (18) is less than the right-hand side:

$$\frac{|U_{gg}|}{2}\left(k + \frac{4\,C(\text{otc})}{|U_{gg}|k}\right)\frac{4\,C(\text{otc})}{|U_{gg}|k} < C(\text{cent}) - C(\text{otc}) \Leftrightarrow 3\,C(\text{otc}) + \frac{8\,C(\text{otc})^2}{|U_{gg}|k^2} < C(\text{cent}),$$

and we are done.

B Non-exclusive participation with heterogeneous capacities

We now consider our special case without exclusivity. We assume that there are two endowment types, $\omega \in \{0, 1\}$, and that there is a continuous uniform distribution over trading capacities $k \in [0, 1]$.

Proposition 5. *There exists $k^\star$ such that banks with capacities $k < k^\star$ participate in the both markets and banks with capacities $k \geq k^\star$ participate only in the OTC market. The post-trade exposures of banks who participate in both markets are equalized $g(\omega, k, \text{otc+cent}) = 1/2$. The post-trade exposures of $\pi = \text{otc}$-banks are*

$$g(0, k, \text{otc}) = \frac{1}{3}\left(1 + k^\star + (k^\star)^2\right),$$
$$g(1, k, \text{otc}) = \frac{1}{3}\left(2 - k^\star - (k^\star)^2\right).$$

where $k^\star$ solves

$$\frac{|U_{gg}|}{2}\left(\frac{1}{2} - g(0, k, \text{otc})\right)\left(\frac{1}{2} + g(0, k, \text{otc}) - (k^\star)^2\right) = C(\text{cent}).$$

$C(\text{cent}) \in \left(0, \frac{5|U_{gg}|}{72}\right)$ *is a necessary condition for this equilibrium to exist (i.e., to make $k^\star$ interior). If, on top of this condition, investors who enter both markets do not have incentive to leave the OTC market, i.e.,*

$$C(\text{otc}) < \frac{|U_{gg}|}{4}\left(\frac{1}{2} - g(0, k, \text{otc})\right)\left(1 - (k^\star)^2\right),$$

then this equilibrium exists. In this equilibrium, the frictional surpluses created by a $\pi = \text{otc}$-bank and by a $\pi = \text{otc+cent}$-bank are

$$\begin{aligned} F(0, k, \text{otc}) &= |U_{gg}|\left(\frac{1}{2} - g(0, k, \text{otc})\right)k^\star\mathbb{E}\left[\max\left\{k, k'\right\}|k' < k^\star\right] \\ &\quad +|U_{gg}|\left(\frac{1}{2} - g(0, k, \text{otc})\right)(1 - k^\star)\,\mathbb{E}\left[\max\left\{k, k'\right\}|k' > k^\star\right] \end{aligned}$$

and

$$F(0,k,\text{otc+cent}) = |U_{gg}| \left(\frac{1}{2} - g(0,k,\text{otc})\right)(1-k^\star)\,\mathbb{E}\left[\max\left\{k,k'\right\}|k' > k^\star\right],$$

respectively.

Proof. Suppose banks with capacities $k < k^\star$ participate in both markets and banks with capacities $k \geq k^\star$ participate only in the OTC market. As a general result, we know post-trade exposures are uniquely determined. By keeping this in mind, we guess and verify that the post-trade exposures of pure OTC banks are independent of k. We start with the banks with endowment of 0:

$$g(0,k,\text{otc}) = \int_0^{k^\star} k dk' + \frac{1}{2}\int_{k^\star}^{1} \max\left\{k,k'\right\} dk' + \frac{1}{2}\int_{k^\star}^{1} \gamma\left[(0,k),(0,k')\right] dk'.$$

Aggregating over $k \geq k^\star$ and using the bilateral feasibility constraint, the post-trade exposure independent of k must be

$$g(0,k,\text{otc}) = k^\star \mathbb{E}\left[k'|k' > k^\star\right] + \frac{1}{2}(1-k^\star)\,\mathbb{E}\left[\max\left\{k,k'\right\}|k,k' > k^\star\right].$$

Here in this case post-trade exposures turn out very simple: The expression reveals that the post-trade exposure of low endowment banks captures the average trade size with the banks with different post-trade exposures. The first term equals the average trade size with the banks who choose to participate in the centralized market, multiplied by the total mass of those banks; and the second term equals the average trade size with the high endowment banks who participate only in the OTC market, multiplied by the total mass of those banks. Using the fact that k is distributed uniformly on $[0,1]$, the expression in the proposition for $g(0,k,\text{otc})$ obtains. By symmetry, $g(1,k,\text{otc}) = 1 - g(0,k,\text{otc})$.

For brevity, let $g = g(0,k,\text{otc})$. Then, Lemma 1 implies the frictional surplus formulas stated in the proposition and that

$$\begin{aligned}
\text{MPV}(0,k,\text{otc}) &= \frac{|U_{gg}|}{2}g^2 + \frac{|U_{gg}|}{2}\left(\frac{1}{2} - g\right)k^\star \mathbb{E}\left[\max\left\{k,k'\right\}|k' < k^\star\right] \\
&+ \frac{|U_{gg}|}{2}\left(\frac{1}{2} - g\right)(1-k^\star)\,\mathbb{E}\left[\max\left\{k,k'\right\}|k' > k^\star\right] \\
\text{MPV}(0,k,\text{cent}) &= \frac{|U_{gg}|}{8} \\
\text{MPV}(0,k,\text{otc+cent}) &= \frac{|U_{gg}|}{8} + \frac{|U_{gg}|}{2}\left(\frac{1}{2} - g\right)(1-k^\star)\,\mathbb{E}\left[\max\left\{k,k'\right\}|k' > k^\star\right].
\end{aligned}$$

Therefore,

$$\text{MPV}(0,k,\text{otc+cent}) - \text{MPV}(0,k,\text{cent}) = \frac{|U_{gg}|}{2}\left(\frac{1}{2} - g\right)(1-k^\star)\,\mathbb{E}\left[\max\left\{k,k'\right\}|k' > k^\star\right],$$

$$\text{MPV}(0, k, \text{otc+cent}) - \text{MPV}(0, k, \text{otc}) = \frac{|U_{gg}|}{2}\left(\frac{1}{2} - g\right)\left(\frac{1}{2} + g - k^\star\right)\mathbb{E}\left[\max\left\{k, k'\right\}|k' < k^\star\right].$$

The equilibrium conditions are

$$\text{MPV}(0, k^\star, \text{otc+cent}) - C(\text{cent}) - C(\text{otc}) = \text{MPV}(0, k^\star, \text{otc}) - C(\text{otc}) > 0$$

and

$$\text{MPV}(0, k^\star, \text{cent}) - C(\text{cent}) < \text{MPV}(0, k^\star, \text{otc+cent}) - C(\text{cent}) - C(\text{otc}),$$

which imply the conditions in the proposition. □

Now we conduct the same thought experiment as in the exclusive case. The effect on social welfare of changing the marginal OTC bank's entry decision to $\pi' = \text{otc+cent}$ is

$$\Delta W = \frac{1}{2}\left[F(0, k^\star, \text{otc+cent}) - F(0, k^\star, \text{otc})\right].$$

The proposition implies

$$\Delta W = -|U_{gg}|\left(\frac{1}{2} - g(0, k, \text{otc})\right)(k^\star)^2,$$

which is negative; i.e., letting the marginal bank enter the centralized market besides the OTC market decreases the social welfare, which implies that there is too much participation in the centralized market. In the exclusivity case, the interpretation of the opposite normative result was based on the match creation-destruction interpretation. In this case, the process of match creation and destruction will not matter because the both scenarios of our thought experiment, otc and otc+cent, feature trading in the OTC market, and hence, the set of OTC matches are exactly the same. However, the composition of match surpluses is different.

In particular, the total frictional surplus a pure OTC agent creates by trading with other OTC participants is larger than the total frictional surplus an OTC+centralized agent creates. This is because a positive-surplus trade between two pure OTC agents has a higher surplus than a trade between one pure OTC and one OTC+centralized agent (i.e., the size of the surplus in the former case is exactly twice the size of the surplus in the latter case, in our symmetric equilibrium with linear marginal benefit). By noting that the half of matches between pure OTC agents are positive-surplus trades, it is easy to see that letting the marginal bank enter the centralized market besides the OTC market will not change the total surplus it creates with pure OTC agents. However, the value of its earlier matches with OTC+centralized agents will be destroyed as it is also an OTC+centralized agent now. Thus, this constitutes the marginal social loss caused by letting it enter the centralized market besides the OTC market.

In the MPV formula, the marginal bank internalizes only the half of the social loss. As it compares this private loss against a "constant" competitive gain, it ends up having too high an incentive to enter the centralized market.

C Capacity constraints in the centralized market

In this appendix, we consider a simple parametric example in which banks face a trading capacity constraint both in the centralized and in the OTC market. Precisely, we follow the analysis of Section 4.2 and assume that banks have heterogeneous endowments uniformly distributed over $[0, 1]$ and homogeneous capacity constraint $k < 1/2$, which applies to both the OTC and the centralized market this time. We assume that participation costs induce exclusive participation either in the OTC or in the centralized market.

Participation equilibrium. Let us guess and verify a participation equilibrium similar to the one considered in Section 4.2, characterized by a threshold $\omega^\star$ such that $\omega^\star + k < 1/2$. Let us focus on the marginal bank. In the OTC market, this bank always trades in the same direction, so its post-trade exposure is

$$g(\omega^\star, k, \text{otc}) = \omega^\star + k$$

However, given the capacity constraint, the bank reaches the same post-trade exposures in the centralized market. Yet, the MPVs are not the same in the two markets: indeed, terms of trade are different. In the centralized market, we take the price to be $U_g(1/2)$.[29] In the OTC market, the average price paid by the marginal bank is:

$$\frac{1}{2}\left[U_g\left(g(\omega^\star, k, \text{cent})\right) + U_g\left(1/2\right)\right],$$

since the average marginal value across all OTC counterparties is $U_g(1/2)$. So one sees that, because of bargaining, the marginal bank buys at higher prices in the OTC market than in the centralized market. The difference between the MPV in the two markets only comes from this term of OTC trade effect, and so is equal to the quantity, k, times the price difference between the centralized and the OTC market, $\frac{1}{2}\left[U_g(g(\omega^\star, k, \text{cent}) - U_g(1/2)\right]$. Therefore, the equilibrium equation for $\omega^\star$ is:

$$\frac{|U_{gg}|}{2} k \left(\frac{1}{2} - \omega^\star - k\right) = C(\text{cent}) - C(\text{otc}).$$

[29] Notice that there are multiple market-clearing prices in the centralized market, since the trading capacity constraint is binding for all banks. However, $U_g(1/2)$ is the only price consistent with symmetric participation incentives, and so with a symmetric participation equilibrium.

It is then straightforward to adapt the argument of Lemma 5 and show that the conjectured participation patterns are optimal.

Welfare analysis. In this special case, we do not need a general envelope analysis of MSV, we can just do the calculations by hand. Suppose that we make the following change to the participation path. We subtract from the OTC market and add to the centralized market, a measure $\varepsilon\, n$, where

$$dn(\omega) = \mathbb{I}_{\{\omega \in [\omega^\star, \omega^\star + \Delta]\}}\, d\omega + \mathbb{I}_{\{\omega \in [1-\omega^\star - \Delta, 1-\omega^\star]\}}\, d\omega,$$

for some small Δ, and some $\varepsilon \to 0$. The change in welfare in the OTC market, relative to autarky, can be calculated using the usual formula.

$$\Delta W(\text{otc}) \simeq -\varepsilon \int \left[\text{MSV}(\omega, k, \text{otc}) - C(\text{otc})\right]\, dn(\omega),$$

where

$$\text{MSV}(\omega, k, \text{otc}) = \text{MPV}(\omega, k, \text{otc}) + \frac{1}{2}\left[F(\omega, k, \text{otc}) - \bar{F}\right].$$

In the centralized market, given that participation is symmetric, one easily sees that post-trade exposures of the participant are given by the same formula $g(\omega) = \omega + k$, i.e., they do not depend on ε. Therefore, the change in welfare, relative to autarky, is simply:

$$\Delta W(\text{cent}) = \varepsilon \left[\int U\left[g(\omega, k, \text{cent})\right]\, dn(\omega) - \int U\left(\omega\right)\, dn(\omega) - C(\text{cent})\right]$$

Because of symmetry, we also have that $P_{\text{cent}} \int \left[g(\omega, k, \text{cent}) - \omega\right]\, dn(\omega) = 0$. Subtracting this from the above expression, we obtain:

$$\Delta W(\text{cent}) = \varepsilon \int \left[\text{MPV}(\omega, k, \text{cent}) - C(\text{cent})\right]\, dn(\omega).$$

Adding up the two and using the marginal condition $\text{MPV}(\omega^\star, k, \text{otc}) - C(\text{otc}) = \text{MPV}(\omega^\star, k, \text{cent}) - C(\text{cent})$, we obtain that, for small Δ:

$$\begin{aligned}\Delta W(\text{otc}) + \Delta W(\text{cent}) &\simeq -\varepsilon \int \left[\text{MSV}(\omega, k, \text{otc}) - \text{MPV}(\omega, k, \text{otc})\right]\, dn(\omega) \\ &= -\frac{\varepsilon}{2}\left[F(\omega, k, \text{otc}) - \bar{F}\right].\end{aligned}$$

We conclude that, in this case, our main formula for welfare analysis remains the same.

D Market resiliency differential

Suppose banks care for the timing of the resolution of uncertainty when they make their participation choice at date $t = 0$. Formally, this implies that the bank's *ex ante* utility is

$$U_0 = \mathbb{E}_0 \left[u_0 \left(\mathbb{E}_{0_+} \left[u_{0_+} \left(W \right) \right] \right) \right],$$

where W is the terminal wealth, $u_{0_+}\left(W\right) = -e^{-\eta W}$, and $u_0\left(\cdot\right)$ is a felicity function that induces a preference for early resolution of uncertainty.

As Van Nieuwerburgh and Veldkamp (2010) explain, banks have a preference for early resolution of uncertainty if and only if $u_0\left(\cdot\right)$ is a convex function. Similar to their Example 2, we pick this convex function to be $u_0\left(z\right) = -log\left(-z\right)$. Combined with the fact that the asset payoff is normally distributed, banks are von Neumann-Morgenstern expected utility maximizers with "mean-variance" felicity functions when they make their participation decisions at date $t = 0$:

$$U_0 = \mathbb{E}_0 \left[\eta \mathbb{E}\left[W\right] - \frac{\eta^2}{2} \mathbb{V}\left[W\right] \right].$$

Equivalently, this expected utility can be specified as

$$\bar{U}_0 = \mathbb{E}_0 \left[\mathbb{E}\left[W\right] - \frac{\eta}{2} \mathbb{V}\left[W\right] \right].$$

Suppose participation costs induce exclusive equilibrium participation. Consider the participation decision of a bank of type (ω, k), where $x = (\omega, k, \text{otc})$ and $x' = (\omega, k, \text{cent})$. Then, the terminal wealth of type x and x' are

$$W\left(x\right) = \begin{cases} -C\left(\text{otc}\right) + g(x)v - \int \gamma(x,x')P_{\text{otc}}(x,x')\, dN(x' \mid X_{\text{otc}}) & \text{if OTC market is open} \\ -C\left(\text{otc}\right) + \omega(x)v & \text{if OTC market shuts down} \end{cases}$$

and

$$W\left(x'\right) = \begin{cases} -C\left(\text{cent}\right) + g(x')v - \varphi(x')P_{\text{cent}} & \text{if the centralized market is open} \\ -C\left(\text{cent}\right) + \omega(x')v & \text{if the centralized market shuts down,} \end{cases}$$

where $g\left(\cdot\right)$ is given by (7). Thus, the expected utility of participating in the OTC market and in the centralized market are

$$\bar{U}_0\left(x\right) = (1-\theta)\left\{ -C\left(\text{otc}\right) + U\left[g(x)\right] - \int \gamma(x,x')P_{\text{otc}}(x,x')\, dN(x' \mid X_{\text{otc}}) \right\} + \theta\left\{ -C\left(\text{otc}\right) + U\left[\omega(x)\right] \right\} \quad (37)$$

and

$$\bar{U}_0\left(x'\right) = (1-\delta\theta)\left\{-C\left(\text{cent}\right) + U\left[g(x')\right] - \varphi(x')P_{\text{cent}}\right\} + \delta\theta\left\{-C\left(\text{cent}\right) + U\left[\omega(x')\right]\right\}, \tag{38}$$

respectively, where $U(g) \equiv \mathbb{E}\left[v\right]g - \frac{\eta}{2}\mathbb{V}\left[v\right]g^2$. After rearranging, (37) and (38) become

$$\bar{U}_0\left(x\right) = -C\left(\text{otc}\right) + U\left[\omega(x)\right] + (1-\theta)\left\{U\left[g(x)\right] - U\left[\omega(x)\right] - \int \gamma(x,x')P_{\text{otc}}(x,x')\,dN(x' \mid X_{\text{otc}})\right\}$$

and

$$\bar{U}_0\left(x'\right) = -C\left(\text{cent}\right) + U\left[\omega(x')\right] + (1-\delta\theta)\left\{U\left[g(x')\right] - U\left[\omega(x')\right] - \varphi(x')P_{\text{cent}}\right\},$$

respectively. Treating these as the certainty-equivalents, the formulas for the marginal private and social values naturally derive from our earlier analysis.