# Appendix

## Firm and Worker Dynamics in a Frictional Labor Market

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This Appendix is organized as follows. Section A studies the case of a hire from unemployment where the internal firm negotiation involves multiple workers. Section B lays out the notation for the fully dynamic model. Section C provides extensive details on the derivation of the joint surplus $\Omega(z, n)$. Section D provides a characterization of the surplus function. Section E derives the limiting behavior of our economy when frictions vanish. Section F details the algorithms used in the paper to compute and estimate the model.

## A *UE hire* when the internal renegotiation involves with multiple workers

In this section, we demonstrate that the case with one worker analyzed in the main text is not a special case and describe internal renegotiation with multiple workers. It is sufficient to consider the case of two incumbent workers, $n = 2$. Without loss of generality, assume that the second worker is paid more than the first, $w_2 > w_1$. As in the approach taken earlier, suppose the firm has posted a vacancy that has met an unemployed worker. We have three cases to consider which illustrate how the firm may use a worker outside the firm to sequentially reduce wages of workers inside the firm.

First, the firm hires *without* renegotiation if:

$$\underbrace{y(z,3) - w_1 - w_2 - b > y(z,2) - w_1 - b}_{\text{No credible threat to } w_2} \quad , \quad \underbrace{y(z,3) - w_1 - w_2 - b > y(z,2) - w_1 - w_2}_{\text{Optimal to hire under } (w_1, w_2)} .$$

Hiring with current wages is preferred to replacing the most expensive incumbent—there is no credible threat—, and given no renegotiation, hiring is optimal. Since $w_2 > w_1$, no credible threat to worker 2 implies no credible threat to worker 1.

Second, the firm hires *with* renegotiation with worker 2 if:

$$\underbrace{y(z,2) - w_1 - b > y(z,3) - w_1 - w_2 - b > y(z,2) - w_2 - b}_{\text{Credible threat for worker 2 only}} \quad , \quad \underbrace{y(z,3) - w_1 - w_2^* - b > y(z,2) - w_1 - w_2^*}_{\text{Optimal to hire under } (w_1, w_2^*)} .$$

The threat is credible for worker 2, but is not for worker 1, and, conditional on renegotiating to $(w_1, w_2^*)$, hiring is optimal.

Third, the firm hires *with* renegotiation with *both* workers if:

$$\underbrace{y(z,2) - w_1 - b > y(z,2) - w_2 - b > y(z,3) - w_1 - w_2 - b}_{\text{Credible threat for both workers}} \quad , \quad \underbrace{y(z,3) - w_1^* - w_2^* - b > y(z,2) - w_1^* - w_2^*}_{\text{Optimal to hire under } (w_1^*, w_2^*)}.$$

In all three cases, the optimal hiring condition can be written as:

$$\Omega(z,3) - \Omega(z,2) > U. \tag{1}$$

This last inequality does not depend on the order of the internal negotiation between firm and workers. In conclusion, the distribution of wages among incumbents again determines the patterns of wage renegotiation, but is immaterial for the sufficient condition for hiring.

Assumption **(A-LC-c)** that was not present in the one worker example plays a role here. Suppose that the renegotiated wage for worker 2 is pushed all the way down to $b$, making her indifferent between staying and quitting. Worker 1 could transfer a negligible amount to worker 2 in exchange of her quitting, which would raise the firm's marginal product and, possibly, remove its own threat. This is problematic for the representation because in this latter case the hiring condition becomes $y(z,2) - y(z,1) - w_1 - b > y(z,1) - w_1$, distinct from (1). Thus, to know whether a firm hires or not, one would need to know the wage distribution inside the firm. **(A-LC-c)** is sufficient to rule out transfers among workers and to prevent this scenario from happening.

Note that, this transfer scheme between workers occurring during the internal negotiation changes the joint value, and hence one can think of **(A-LC-c)** as being subsumed into **(A-IN)** already.

# B Notation for dynamic model

We first specify the value function of an individual worker $i$ in a firm with arbitrary state $x$: $V(x,i)$. We then specify the value function of the firm: $J(x)$. Combining all workers' value functions with that of the firm we define the joint value: $\Omega(x)$. We then apply the assumptions from Section **??** which allow us to reduce $(x)$ to only the number of workers and productivity of the firm, $(z,n)$. Finally we take the continuous work force limit to derive a Hamilton-Jacobi-Bellman (HJB) equation for $\Omega(z,n)$ Applying the definition of total surplus used above, we obtain a HJB equation in $S(z,n)$ which we use to construct the equilibrium.

### B.1 Worker value function: $V$

As in the static example, let $U$ be the value of unemployment. It is convenient to define separately worker $i$'s value when employed at firm $x$ *before* the quit, layoff and exit decisions, $\boldsymbol{V}(x,i)$, and their value *after* these decisions, $V(x,i)$.[1]

**Value of unemployment.** Let $h_U(x)$ denote how the state of firm $x$ is updated when it hires an unemployed worker.[2] Let $\mathcal{A}$ denote the set of firms making job offers that an unemployed worker would accept. The value of unemployment $U$ therefore satisfies

$$\rho U = b + \lambda^U(\theta) \int_{x \in \mathcal{A}} \left[\boldsymbol{V}\left(h_U(x), i\right) - U\right] dH_v(x)$$

where $H_v$ is the vacancy-weighted distribution of firms. If $x \notin \mathcal{A}$, then the worker remains unemployed.

**Stage I.** To relate the value of the worker pre separation, $\boldsymbol{V}(x,i)$, to that post separation, $V(x,i)$, we require the following notation regarding firm and co-worker actions. Since workers do not form 'unions' within the firm, all of these actions are taken as given by worker $i$.

- Let $\epsilon(x) \in \{0,1\}$ denote the exit decision of firm, and $\mathcal{E} = \{x : \epsilon(x) = 1\}$ the set of $x$'s for which the firm exits.

- Let $\ell(x) \in \{0,1\}^{n(x)}$ be a vector of zeros and ones of length $n(x)$, with generic entry $\ell_i(x)$, that characterizes the firm's decision to lay off incumbent worker $i \in \{1, \ldots, n(x)\}$, and $\mathcal{L} = \{(x,i) : \ell_i(x) = 1\}$ the set of $(x,i)$ such that worker $(x,i)$ is laid off.

- Let $q^U(x) \in \{0,1\}^{n(x)}$ be a vector of length $n(x)$, with generic entry $q_i^U(x)$ that characterizes an incumbent workers' decisions to quit, and $\mathcal{Q}^U = \left\{(x,i) : q_i^U(x) = 1\right\}$ the set of $(x,i)$ such that worker $(x,i)$ quits into unemployment.

- Let $\kappa(x) = (1 - \boldsymbol{\ell}(x)) \circ (1 - q_U(x))$ be an element-wise product vector that identifies workers that are kept in the firm, and $\mathcal{S} = \mathcal{L} \cup \mathcal{Q}^U = \{(x,i) : \kappa_i(x) = 0\}$, the set of $(x,i)$ such that worker $(x,i)$ separates into unemployment.

- Let $s(x, \kappa(x))$ denote how the state of firm $x$ is updated when workers identified by $\kappa(x)$ are kept. This includes any renegotiation.

[1] In terms of Figure **??**, the value $\boldsymbol{V}$ is computed after the first stage of the flow chart, and the value $V$ after the second stage, in the case that the firm stays in operation.

[2] For example, size would be update from $n$ to $n+1$ and possibly some of the incumbent wages would be bargained down.

Given these sets and functions, the pre separation value $\boldsymbol{V}(x,i)$ satisfies:

$$\boldsymbol{V}(x,i) = \underbrace{\epsilon(x)U}_{\text{Exit}} + (1-\epsilon(x))\Big[\underbrace{\mathbb{I}_{\{(x,i)\notin\mathcal{S}\}}V(s(x,\kappa(x)),i)}_{\text{Continuing employment}} + \underbrace{\mathbb{I}_{\{(x,i)\in\mathcal{S}\}}U}_{\text{Separations and Quits}}\Big]$$

**Stage II.** It is helpful to characterize the value of employment post separation decisions, $V(x,i)$, in terms of the three distinct types of events described in Figure **??**. First, the value changes due to *'Direct'* labor markets shocks to worker $i$, $V_D(x,i)$. These include her match being destroyed exogenously or meeting a new potential employer. Second, the value changes due to labor market shocks hitting other workers in the firm, $V_I(x,i)$, including their matches being exogenously destroyed or them meeting new potential employers. These events have an *'Indirect'* impact on worker $i$. Third, the value changes due to events on the *'Firm'* side, $V_F(x,i)$, including the firm contacting new workers and receiving productivity shocks. Combining events and exploiting the fact that in continuous time they are mutually exclusive, we obtain the following, where $w(x,i)$ is the wage paid to worker $i$:

$$\rho V(x,i) = w(x,i) + \rho V_D(x,i) + \rho V_I(x,i) + \rho V_F(x,i).$$

We note that the wage function $w(x,i)$ includes the transfers between worker $i$ and the firm that may occur at the stage of vacancy posting (after separations and before the labor market opens), as discussed in Section **??** in the context of the static example. These transfers can depend on the entire wage distribution inside the firm which is subsumed in the state vector $x$.

**Direct events.** We first characterize changes in value due to labor market shocks directly to worker $i$ in firm $x$, $V_D(x,i)$. Exogenous separation shocks arrive at rate $\delta$ and draws of outside offers arrive at rate $\lambda^E(\theta)$ from the vacancy-weighted distribution of firms $H_v$. If worker $i$ receives a sufficiently good outside offer from $x'$, she quits to the new firm. We denote by $\mathcal{Q}^E(x,i)$ the set of such quit-firms $x'$ for $i$. Otherwise, the worker remains with the current firm but with an updated contract. Therefore $V_D(x,i)$ satisfies

$$\begin{aligned}\rho V_D(x,i) = \underbrace{\delta[U - V(x,i)]}_{\text{Exogenous separation}} &+ \lambda^E(\theta)\underbrace{\int_{x'\in\mathcal{Q}^E(x,i)}\left[\boldsymbol{V}\left(h_E(x,i,x'),i\right) - V(x,i)\right]dH_v(x')}_{EE\ \text{Quit}} \\ &+ \lambda^E(\theta)\underbrace{\int_{x'\notin\mathcal{Q}^E(x,i)}\left[\boldsymbol{V}\left(r(x,i,x'),i\right) - V(x,i)\right]dH_v(x')}_{\text{Retention}},\end{aligned}$$

where $h_E(x,i,x')$ describes how the state of a poaching firm $x'$ gets updated when it hires worker $i$ from firm $x$. Similarly, $r(x,i,x')$ updates $x$ when—after meeting firm $x'$—worker $i$ in firm $x$ is retained and renegotiates its value. In all functions with three arguments $(x,i,x')$, the first argument denotes the origin firm, the second identifies the worker, and the third the potential destination firm.

**Indirect events.** We next characterize changes in value due to the same labor market shocks hitting other workers in firm $x$, $V_I(x,i)$. The value $V_I(x,i)$ satisfies

$$\begin{aligned}\rho V_I(x,i) = \sum_{j\neq i}^{n(x)} \Bigg\{ & \underbrace{\delta\left[\boldsymbol{V}(d(x,j),i) - V(x,i)\right]}_{\text{Exogenous separation}} + \lambda^E(\theta) \underbrace{\int_{x'\in\mathcal{Q}^E(x,j)} \left[\boldsymbol{V}\left(q_E(x,j,x'),i\right) - V(x,i)\right] dH_v(x')}_{EE \text{ Quit}} \\ & + \lambda^E(\theta) \underbrace{\int_{x'\notin\mathcal{Q}^E(x,j)} \left[\boldsymbol{V}\left(r(x,j,x'),i\right) - V(x,i)\right] dH_v(x')}_{\text{Retention}} \Bigg\},\end{aligned}$$

where $d(x,j)$ updates $x$ when worker $j$ exogenously separates, and $q_E(x,j,x')$ when worker $j$ quits to firm $x'$.

**Firm events.** Finally, we characterize changes in value due to events that directly impact the firm and hence indirectly its workers, $V_F(x,i)$. Taking as given the firm's vacancy posting policy $v(x)$ and other actions, $V_F(x,i)$ satisfies

$$\begin{aligned}\rho V_F(x,i) &= \\ UE \text{ Hire} \quad & \phi q(\theta) v(x)\left[\boldsymbol{V}\left(h_U(x),i\right) - V(x,i)\right]\cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\ UE \text{ Threat} \quad & +\phi q(\theta) v(x)\left[\boldsymbol{V}\left(t_U(x),i\right) - V(x,i)\right]\cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\ EE \text{ Hire} \quad & +(1-\phi)\, q(\theta) v(x) \int_{x\in\mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(h_E(x',i',x),i\right) - V(x,i)\right] dH_n(x',i') \\ EE \text{ Threat} \quad & +(1-\phi)\, q(\theta) v(x) \int_{x\notin\mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(t_E(x',i',x),i\right) - V(x,i)\right] dH_n(x',i') \\ \text{Shock} \quad & +\Gamma_z\left[\boldsymbol{V}, V\right](x,i)\end{aligned}$$

where $t_U(x)$ updates $x$ when an unemployed worker is met and not hired, but could be possibly used as a threat in firm $x$. Similarly, $t_E(x',i',x)$ updates $x$ when worker $i'$ employed at firm $x'$ is met, not hired, but could be used as a threat. And, with a slight abuse of notation, $H_n(x',i')$ gives the joint distribution of firms $x'$ and worker types within firms $i'$.

Finally, $\Gamma_z[\boldsymbol{V}, V](x,i)$ identifies the contribution of productivity shocks $z$ to the Bellman equation.

At this stage we only require that the productivity process is Markovian with an infinitesimal generator. Later we will specialize this to a diffusion process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$ such that

$$\begin{aligned} \Gamma_z\left[\boldsymbol{V}, V\right](x,i) &= \mu(z) \lim_{dz\to 0} \frac{\boldsymbol{V}\left((x,z+dz),i\right) - V\left(x,z,i\right)}{dz} \\ &+ \frac{\sigma^2(z)}{2} \lim_{dz\to 0} \frac{\boldsymbol{V}\left((x,z+dz),i\right) + \boldsymbol{V}\left((x,z-dz),i\right) - 2V\left(x,z,i\right)}{dz^2} \end{aligned} \tag{2}$$

In the case that $\boldsymbol{V} = V$, this becomes the standard expression for a diffusion featuring the first and second derivatives of $V$ with respect to $z$: $\Gamma_z[V](x,i) = \mu(z)V_z(x,z,i) + \frac{1}{2}\sigma(z)^2 V_{zz}(x,z,i)$.[3]

In the event productivity changes or $n(x)$ changes because of exogenous labor market events, the worker will want to reassess whether to stay with the firm or not. Additionally, the firm may want to reassess whether to exit or fire some workers. Bold values $\boldsymbol{V}$ capture any case where the state changes.

### B.2 Firm value function: $J$

Consistent with the notation we used for workers' values, let $\boldsymbol{J}(x)$ and $J(x)$ be the values of the firm at the corresponding points of an interval $dt$. For now, we take the vacancy creation decision $v(x)$ as given. At the end of the section we describe the expected value of an entrant firm.

**Stage I.** Consistent with the first stage worker value function, we define the firm value before the exit/layoff/quit decision, where we recall that $\vartheta$ is the firm's value of exit, or scrap value:

$$\boldsymbol{J}(x) = \epsilon(x)\,\vartheta + [1-\epsilon(x)]\,J\left(s\left(x,\kappa(x)\right)\right).$$

**Stage II.** Given a vacancy policy $v(x)$, let $J(x)$ be the value of a firm with state $x$ *after* the layoff/quit, exit. It is convenient to split the value of the firm, as we did for the worker, into three components

$$\rho J(x) = \underbrace{y(x) - \sum_{i=1}^{n(x)} w_i(x,i)}_{\text{Flow profits}} + \underbrace{\rho J_W(x)}_{\text{Workforce events}} + \underbrace{\rho J_F(x) - c\left(v(x),x\right)}_{\text{Firm events net of vacancy costs}}.$$

For a given policy $v(x)$ there is a set of associated transfers between workers and the firm which, as for the worker value function, are implicit in the wage function $w(x,i)$.

[3] Note that in (2) we abuse notation and write the state as $(x,z)$ with some redundancy since $z$ is clearly a member of $x$. We also note that we are not constrained to a diffusion process. We could also consider a Poisson process where, at exogenous rate $\eta$, $z$ jumps according to the transition density $\Pi(z,z')$: $\Gamma_z\left[\boldsymbol{V}, V\right](x,i) = \eta[\sum_{z'\in Z} \boldsymbol{V}\left((x,z'),i\right)\Pi(z',z) - V(x,z,i)]$.

The component $J_W(x)$ is given by

$$\begin{aligned}
\rho J_W(x) &= \\
\text{Destruction} \quad & \delta \sum_{i=1}^{n(x)} \left[\boldsymbol{J}(d(x,i)) - J(x)\right] \\
EE\text{ Quit} \quad & + \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) - J(x)\right] dH_v\left(x'\right) \\
\text{Retention} \quad & + \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(r\left(x,i,x'\right)\right) - J(x)\right] dH_v\left(x'\right).
\end{aligned}$$

The component $J_F(x)$ is given by

$$\begin{aligned}
\rho J_F(x) &= \\
UE\text{ Hire} \quad & \phi q(\theta) v(x) \left[\boldsymbol{J}(h_U(x)) - J(x)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE\text{ Threat} \quad & + \phi q(\theta) v(x) \left[\boldsymbol{J}(t_U(x)) - J(x)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE\text{ Hire} \quad & + (1-\phi) q(\theta) v(x) \int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) - J(x)\right] dH_n\left(x',i'\right) \\
EE\text{ Threat} \quad & + (1-\phi) q(\theta) v(x) \int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(t_E\left(x',i',x\right)\right) - J(x)\right] dH_n\left(x',i'\right) \\
\text{Shock} \quad & + \Gamma_z\left[\boldsymbol{J}, J\right](x)
\end{aligned}$$

It is useful to recall that, in continuous time at most one contact is made per instant. That is, either one worker is exogenously separated, or one worker is contacted by another firm, or one worker is met by posting vacancies (at rate $q(\theta)v(x)$), or a shock hits the firm. Note also that we have bold $\boldsymbol{J}$'s in each line since after any of these events, the firm may want to layoff some workers or exit, and workers may want to quit.

**Entry.** The expected value of an entrant firm is

$$J_0 = -c_0 + \int \boldsymbol{J}(x_0)\, d\Pi_0(z_0) \tag{3}$$

where $x_0$ is the state of the entrant firm which includes only the random productivity value $z_0$ drawn from $\Pi_0$ since we assumed the initial number of workers is 0. The argument of the integral is $\boldsymbol{J}$, which incorporates the firm's decision to exit or operate after observing $z_0$. Entry occurs when $J_0 > 0$.

# C Derivation of the joint value function $\Omega$

We define the **joint value** of the firm and its employed workers $\Omega(x) := J(x) + \sum_{i=1}^{n(x)} V(x,i)$. We also define the joint value before exit/quit/layoff decisions: $\boldsymbol{\Omega}(x) := \boldsymbol{J}(x) + \sum_{i=1}^{n(x)} \boldsymbol{V}(x,i)$.

## C.1 Combinining worker and firm values

In this section, we show that summing firm and worker values, then applying these definitions delivers the following Bellman equation for the joint value:

$$
\begin{aligned}
\rho\Omega(x) &= y(x) - c(v(x), x) && (4)\\
\text{Destruction} &\quad + \sum_{i=1}^{n(x)} \delta\left[\boldsymbol{\Omega}(d(x,i)) + U - \Omega(x)\right]\\
\text{Retention} &\quad + \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) - \Omega(x)\right] dH_v\left(x'\right)\\
EE\text{ Quit} &\quad + \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(q_E\left(x,i,x'\right)\right) + \boldsymbol{V}\left(h_E\left(x,i,x'\right), i\right) - \Omega(x)\right] dH_v\left(x'\right)\\
UE\text{ Hire} &\quad + \phi q(\theta) v(x)\left[\boldsymbol{\Omega}(h_U(x)) - U - \Omega(x)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}\\
UE\text{ Threat} &\quad + \phi q(\theta) v(x)\left[\boldsymbol{\Omega}(t_U(x)) - \Omega(x)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}}\\
EE\text{ Hire} &\quad + (1-\phi) q(\theta) v(x) \int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \boldsymbol{V}\left(h_E\left(x',i',x\right), i'\right) - \Omega(x)\right] dH_n\left(x',i'\right)\\
EE\text{ Threat} &\quad + (1-\phi) q(\theta) v(x) \int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(t_E\left(x',i',x\right)\right) - \Omega(x)\right] dH_n\left(x',i'\right)\\
\text{Shock} &\quad + \Gamma_z\left[\boldsymbol{\Omega}, \Omega\right](x).
\end{aligned}
$$

Note that this joint value is only written in terms of other joint values and worker values. However, it involves both firm and worker decisions through the sets $\mathcal{A}$, $\mathcal{Q}^E$ and the vacancy policy, $v(x)$.

**Derivation.** We start by computing the sum of the workers' values at a particular firm. Summing values of all the employed workers

$$
\begin{aligned}
\rho \sum_{i=1}^{n(x)} V(x,i) &= \sum_{i=1}^{n(x)} w(x,i) \\
\text{Destructions} &\quad + \sum_{i=1}^{n(x)} \delta \left[U - V(x,i)\right] \\
\text{Retentions} &\quad + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{V}\left(r\left(x,i,x'\right),i\right) - V(x,i)\right] dH_v\left(x'\right) \\
EE \text{ Quits} &\quad + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{V}\left(h_E\left(x,i,x'\right)\right) - V(x,i)\right] dH_v\left(x'\right) \\
\text{Incumbents} &\quad + \sum_{i=1}^{n(x)} \rho V_I(x,i) \\
\text{Firm} &\quad + \sum_{i=1}^{n(x)} \rho V_D(x,i)
\end{aligned}
$$

where the indirect term due to incumbents can be written as:

$$
\begin{aligned}
\sum_{i=1}^{n(x)} \rho V_I(x,i) &= \\
\text{Destructions} &\quad \sum_{i=1}^{n(x)} \sum_{j \neq i}^{n(x)} \delta \left[\boldsymbol{V}\left(d(x,j),i\right) - V(x,i)\right] \\
\text{Retentions} &\quad + \sum_{i=1}^{n(x)} \sum_{j \neq i}^{n(x)} \lambda^E \int_{x' \notin \mathcal{Q}^E(x,j)} \left[\boldsymbol{V}\left(r\left(x,j,x'\right),i\right) - V(x,i)\right] dH_v\left(x'\right) \\
EE \text{ Quits} &\quad + \sum_{i=1}^{n(x)} \sum_{j \neq i}^{n(x)} \lambda^E \int_{x' \in \mathcal{Q}^E(x,j)} \left[\boldsymbol{V}\left(q_E\left(x,j,x'\right),i\right) - V(x,i)\right] dH_v\left(x'\right)
\end{aligned}
$$

and the indirect term due to the firm can be written as:

$$
\begin{aligned}
\sum_{i=1}^{n(x)} \rho V_F(x,i) &= \\
UE \text{ Hires} \quad & qv\left(x\right)\phi \sum_{i=1}^{n(x)} \left[\boldsymbol{V}\left(h_U\left(x\right),i\right) - V\left(x,i\right)\right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE \text{ Threats} \quad & +qv\left(x\right)\phi \sum_{i=1}^{n(x)} \left[\boldsymbol{V}\left(t_U\left(x\right),i\right) - V\left(x,i\right)\right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE \text{ Hires} \quad & +qv\left(x\right)\left(1-\phi\right) \sum_{i=1}^{n(x)} \int_{x\in\mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) - V\left(x,i\right)\right] dH_n\left(x',i'\right) \\
EE \text{ Threats} \quad & +qv\left(x\right)\left(1-\phi\right) \sum_{i=1}^{n(x)} \int_{x\notin\mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - V\left(x,i\right)\right] dH_n\left(x',i'\right) \\
\text{Shocks} \quad & +\sum_{i=1}^{n(x)} \Gamma_z[\boldsymbol{V},V](x,i)
\end{aligned}
$$

We now collect terms.

**Destructions.** When worker $i$ separates from firm $x$, the sum of the changes in values of all employed workers at its own firm is given by:

$$
\begin{aligned}
\text{Destructions} &= \delta\left[U - V\left(x,i\right)\right] + \delta \sum_{j\neq i}^{n(x)} \left[\boldsymbol{V}\left(d(x,i),j\right) - V\left(x,j\right)\right] \\
&= \delta\left[U + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(d(x,i),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right]
\end{aligned}
$$

**Retentions.** When $i$ renegotiates at firm $x$, the sum of the changes in values of all employed workers at its own firm is given by:

$$
\begin{aligned}
\text{Retentions} &= \lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(r\left(x,i,x'\right),i\right) - V\left(x,i\right)\right] dH_v\left(x'\right) \\
&\quad +\lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \sum_{j\neq i}^{n(x)} \left[\mathbf{V}\left(r\left(x,i,x'\right),j\right) - V\left(x,j\right)\right] dH_v\left(x'\right) \\
&= \lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(r\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dH_v\left(x'\right) \\
&= \lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\sum_{j=1}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dH_v\left(x'\right)
\end{aligned}
$$

**Quits.** Similarly, when $i$ quits firm $x$, the sum of the changes in values of all employed workers at its own firm is given by:

$$EE \text{ Quits} = \lambda^E \int_{x' \in Q(x,i)} \left[ \boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right) \right] dH_v\left(x'\right)$$

**Combining terms.** Before summing up all these terms, define for convenience the total worker value:

$$\begin{aligned}
\rho \overline{V}(x) \quad &= \quad \sum_{i=1}^{n(x)} w(x,i) \\
\text{Destructions} \quad & \quad + \sum_{i=1}^{n(x)} \delta \left[ U + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(d(x,i),j\right) - \sum_{j=1}^{n(x)} V(x,j) \right] \\
\text{Retentions} \quad & \quad + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[ \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V(x,j) \right] dH_v\left(x'\right) \\
EE \text{ Quits} \quad & \quad + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[ \boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V(x,j) \right] dH_v\left(x'\right) \\
UE \text{ Hires} \quad & \quad + qv(x)\,\phi \sum_{i=1}^{n(x)} \left[ \boldsymbol{V}\left(h_U(x),i\right) - V(x,i) \right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE \text{ Threats} \quad & \quad + qv(x)\,\phi \sum_{i=1}^{n(x)} \left[ \boldsymbol{V}\left(t_U(x),i\right) - V(x,i) \right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE \text{ Hires} \quad & \quad + qv(x)\,(1-\phi) \sum_{i=1}^{n(x)} \int_{x \in \mathcal{Q}^E(x',i')} \left[ \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) - V(x,i) \right] dH_n\left(x',i'\right) \\
EE \text{ Threats} \quad & \quad + qv(x)\,(1-\phi) \sum_{i=1}^{n(x)} \int_{x \notin \mathcal{Q}^E(x',i')} \left[ \boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - V(x,i) \right] dH_n\left(x',i'\right) \\
\text{Shocks} \quad & \quad + \sum_{i=1}^{n(x)} \Gamma_z[\boldsymbol{V}, V](x,i)
\end{aligned}$$

Now sum, up all the previous terms, collect terms and use the definition of $\overline{V}(x)$:

$$
\begin{array}{rl}
\rho\overline{V}(x) = & \sum_{i=1}^{n(x)} w(x,i) \\
\text{Destructions} & + \sum_{i=1}^{n(x)} \delta \left[ U + \sum_{j\neq i}^{n(x)} \boldsymbol{V}(d(x,i),j) - \overline{V}(x) \right] \\
\text{Retentions} & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[ \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right) - \overline{V}(x) \right] dH_v\left(x'\right) \\
EE \text{ Quits} & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[ \boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - \overline{V}(x) \right] dH_v\left(x'\right) \\
UE \text{ Hires} & + qv(x)\,\phi \left[ \sum_{i=1}^{n(x)} \boldsymbol{V}(h_U(x),i) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE \text{ Threats} & + qv(x)\,\phi \left[ \sum_{i=1}^{n(x)} \boldsymbol{V}(t_U(x),i) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE \text{ Hires} & + qv(x)(1-\phi) \int_{x \in \mathcal{Q}^E(x',i')} \left[ \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) - \overline{V}(x) \right] dH_n\left(x',i'\right) \\
EE \text{ Threats} & + qv(x)(1-\phi) \int_{x \notin \mathcal{Q}^E(x',i')} \left[ \sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - \overline{V}(x) \right] dH_n\left(x',i'\right) \\
\text{Shocks} & + \Gamma_z[\overline{\boldsymbol{V}},\overline{V}](x)
\end{array}
$$

Adding this last equation to the Bellman equation for $J(x)$ yields

$$
\begin{array}{rcl}
\rho\Omega(x) & = & y(x) - c(v(x), x) \\
\text{Destructions} & & + \sum_{i=1}^{n(x)} \delta \left[ \boldsymbol{J}(d(x,i)) + U + \sum_{j \neq i}^{n(x)} \boldsymbol{V}(d(x,i), j) - J(x) - \overline{V}(x) \right] \\
\text{Retentions} & & +\lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[ \boldsymbol{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right), j\right) - J(x) - \overline{V}(x) \right] dH_v\left(x'\right) \\
EE \text{ Quits} & & +\lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[ \boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) + \boldsymbol{V}\left(h_E\left(x,i,x'\right), i\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right), j\right) - J(x) - \overline{V}(x) \right] dH_v\left(x'\right) \\
UE \text{ Hires} & & +qv(x)\phi \left[ \boldsymbol{J}(h_U(x)) + \sum_{i=1}^{n(x)} \boldsymbol{V}(h_U(x), i) - J(x) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE \text{ Threats} & & +qv(x)\phi \left[ \boldsymbol{J}(t_U(x)) + \sum_{i=1}^{n(x)} \boldsymbol{V}(t_U(x), i) - J(x) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE \text{ Hires} & & +qv(x)(1-\phi) \int_{x \in \mathcal{Q}^E(x',i')} \left[ \boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right), i\right) - J(x) - \overline{V}(x) \right] dH_n\left(x',i'\right) \\
EE \text{ Threats} & & +qv(x)(1-\phi) \int_{x \notin \mathcal{Q}^E(x',i')} \left[ \boldsymbol{J}\left(t_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_E\left(x',i',x\right), i\right) - J(x) - \overline{V}(x) \right] dH_n\left(x',i'\right) \\
\text{Shocks} & & +\Gamma_z[\boldsymbol{J} + \overline{\boldsymbol{V}}, J + \overline{V}](x) - J(x) - \overline{V}(x)
\end{array}
$$

Collecting terms and using the definition of $\Omega$ :

$$
\begin{array}{rcl}
\rho\Omega(x) & = & y(x) - c(v(x), x) \\
\text{Destructions} & & + \sum_{i=1}^{n(x)} \delta \left[ \boldsymbol{\Omega}(d(x,i)) + U - \Omega(x) \right] \\
\text{Retentions} & & +\lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[ \boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) - \Omega(x) \right] dH_v\left(x'\right) \\
EE \text{ Quits} & & +\lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[ \boldsymbol{\Omega}\left(q_E\left(x,i,x'\right)\right) + \boldsymbol{V}\left(h_E\left(x,i,x'\right), i\right) - \Omega(x) \right] dH_v\left(x'\right) \\
UE \text{ Hires} & & +qv(x)\phi \left[ \boldsymbol{\Omega}(h_U(x)) - U - \Omega(x) \right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE \text{ Threats} & & +qv(x)\phi \left[ \boldsymbol{\Omega}(t_U(x)) - \Omega(x) \right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE \text{ Hires} & & +qv(x)(1-\phi) \int_{x \in \mathcal{Q}^E(x',i')} \left[ \boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \boldsymbol{V}\left(h_E\left(x',i',x\right), i'\right) - \Omega(x) \right] dH_n\left(x',i'\right) \\
EE \text{ Threats} & & +qv(x)(1-\phi) \int_{x \notin \mathcal{Q}^E(x',i')} \left[ \boldsymbol{\Omega}\left(t_E\left(x',i',x\right)\right) - \Omega(x) \right] dH_n\left(x',i'\right) \\
\text{Shocks} & & +\Gamma_z[\overline{\boldsymbol{\Omega}}, \overline{\Omega}](x)
\end{array}
$$

## C.2 Value sharing

To make progress on (4), we begin by stating seven intermediate results, conditions **(C-RT)-(C-E)** which we prove from the assumptions listed in Section **??**. These results establish how worker values $\boldsymbol{V}$ in (4) evolve in the six cases of hiring, retention, layoff, quits, exit and vacancy creation. Next, we apply conditions **(C-RT)-(C-E)** to (4).

To highlight the structure of the argument, we note a key implication our zero-sum game assumption **(A-IN)**: during internal negotiation, any value lost to one party must accrue to the other. This feature is obvious in the static model, and extends readily to our dynamic environment. In other words, the joint value of the firm plus its incumbent workers is invariant during the negotiation. We use this property extensively in the proof. This generalizes pairwise efficient bargaining—commonly used in one-worker firm models with linear production—to an environment with multi-worker firms and decreasing returns in production.

We now state the seven conditions that we apply to (4). In section C.3 below, we prove how each of them is implied by the assumptions of Section **??**.

**(C-RT)** **Retentions and Threats.** First, if firm $x$ meets an unemployed worker and the worker is not hired but only used as a threat, then the joint value of coalition $x$ does not change since threats only redistribute value within the coalition. Second, when firm $x$ uses employed worker $i'$ from firm $x'$ as a threat, the joint value of coalition $x$ does not change. Third, when firm $x$ meets worker $i'$ at $x'$ and the worker is retained by firm $x'$, the joint value of coalition $x'$ does not change. Formally,

$$\boldsymbol{\Omega}\left(r\left(x', i', x\right)\right) = \Omega(x') \quad , \quad \boldsymbol{\Omega}\left(t_U\left(x\right)\right) = \Omega(x) \quad , \quad \boldsymbol{\Omega}\left(t_E\left(x', i', x\right)\right) = \Omega(x).$$

Respectively, these imply that the *Retention*, *UE Threat* and *EE Threat* components of (4) are equal to zero.

**(C-UE)** *UE* **Hires.** An unemployed worker that meets firm $x$ is hired when $x \in \mathcal{A}$. This set consists of firms that have a joint value after hiring that is higher than the pre-hire joint value plus the outside value of the hired worker. Due to the take-leave offer, the new hire receives her outside value, which is the value of unemployment:

$$\mathcal{A} = \{x | \boldsymbol{\Omega}(h_U(x)) - \Omega(x) \geq U\} \quad , \quad \boldsymbol{V}\left(h_U\left(x\right), i\right) = U.$$

**(C-EE)** *EE* **Hires.** An employed worker $i'$ at firm $x'$ that meets firm $x$ is hired when $x \in \mathcal{Q}^E\left(x', i'\right)$. This set consists of firms that have a higher marginal joint value than that of the current firm:

$$\mathcal{Q}^E\left(x',i'\right)=\left\{x\middle|\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right)-\Omega\left(x\right)\geq\Omega(x')-\boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)\right\}.$$

Due to the take-leave offer, the new hire receives her outside value, which is the marginal joint value at her current firm:

$$\boldsymbol{V}\left(h_E\left(x',i',x\right)\right)=\Omega\left(x'\right)-\boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right).$$

**(C-EU)** *EU* **Quits and Layoffs.** An employed worker $i$ at firm $x$ quits to unemployment when $(x,i)\in\mathcal{Q}^U$. This set consist of states $x$ such that the marginal joint value is less than the value of unemployment:

$$\begin{aligned}\mathcal{Q}^U &= \left\{(x,i)\middle|\Omega\left(\widehat{s}_{q1}\left(x,i\right)\right)+U>\Omega\left(\widehat{s}_{q0}\left(x,i\right)\right)\right\},\\ \text{where}\quad \widehat{s}_{q1}\left(x,i\right) &= s\left(x,\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=1\right]\right)\circ\left(1-\ell\left(x\right)\right)\right),\\ \widehat{s}_{q0}\left(x,i\right) &= s\left(x,\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=0\right]\right)\circ\left(1-\ell\left(x\right)\right)\right).\end{aligned}$$

The first expression captures when worker $i$ quits, and the second where worker $i$ does not. Similarly, an *EU* layoff will be chosen by the firm when $(x,i)\in\mathcal{L}$:

$$\begin{aligned}\mathcal{L} &= \left\{(x,i)\middle|\Omega\left(\widehat{s}_{\ell1}\left(x,i\right)\right)+U>\Omega\left(\widehat{s}_{\ell0}\left(x,i\right)\right)\right\},\\ \text{where}\quad \widehat{s}_{\ell1}\left(x,i\right) &= s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=1\right]\right)\circ\left(1-q_U\left(x\right)\right)\right),\\ \widehat{s}_{\ell0}\left(x,i\right) &= s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=0\right]\right)\circ\left(1-q_U\left(x\right)\right)\right).\end{aligned}$$

The first expression captures when worker $i$ is laid off, and the second when worker $i$ is not.

**(C-X)** **Exit.** A firm $x$ exits when $x\in\mathcal{E}$. This set consists of the states in which the total outside value of the firm and its workers is larger than the joint value of operation:

$$\mathcal{E}=\left\{x\middle|\vartheta+n\left(s\left(x,\kappa\left(x\right)\right)\right)\cdot U>\Omega\left(s\left(x,\kappa\left(x\right)\right)\right)\right\}.$$

**(C-V)** **Vacancies.** The expected return to a matched vacancy $R(x)$ depends only on the joint value, and so the firm's optimal vacancy policy $v(x)$ depends only on the joint value. The policy $v(x)$ solves

$$\max_v\ q(\theta)vR(x)-c\left(v,x\right),$$

where the expected return to a matched vacancy is

$$
\begin{aligned}
R(x) \quad = \quad & \phi \underbrace{\left[\boldsymbol{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) - U\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}}_{\text{Return from unemployed worker match}} \\
+ \quad & (1-\phi) \underbrace{\int_{x \in \mathcal{Q}^E(x',i')} \left\{ \left[\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right)\right] - \left[\Omega\left(x'\right) - \boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)\right] \right\} dH_n\left(x',i'\right)}_{\text{Expected return from employed worker match}}.
\end{aligned}
$$

**(C-E) Entry.** A firm enters if and only if

$$\int \boldsymbol{\Omega}\left(x_0\right) d\Pi_0(z) \geq c_0 + n_0 U.$$

**Summarizing (C).** The substantive result is that all firm and worker decisions and employed workers' values can be expressed in terms of joint value $\Omega$ and exogenous worker outside option $U$.

## C.3 Proof of Conditions (C)

### C.3.1 Proof of C-UE and C-RT ($UE$ Hires and $UE$ Threats)

In this subsection, we consider a meeting between a firm $x$ and an unemployed worker. Following **A-IN** and **A-EN**, the firm internally renegotiates according to a zero-sum game with its incumbent workers and makes a take-leave offer to the new worker. Intuitively, having the worker "at the door" is identical to having her hired at value $U$ for the firm and for all incumbent workers: the firm can always make new take-leave offers to its incumbents after hiring the new worker. Hence, we expect the firm to make one take-leave offer to the new worker and its incumbents at the time of the meeting, and not make a new, different offer to is incumbents afer hiring has taken place.

We start by showing this equivalence formally. To do so, when meeting an unemployed worker, we let the firm conduct internal renegotiation with its incumbent workers and make an offer to the new worker. Then, we let a second round of internal offers take place after the hiring. We introduce some notation to keep track of values throughout the internal and external negotiations. To fix ideas, we denote by (IR1) the first round of internal negotiation, pre-external negotiation. We denote by (IR2) the second round of internal negotiation, post-hire.

Post-hire and post-internal negotiation (IR2) values are denoted with double stars. Post-internal-negotiation (IR1) but pre-external-negotiation values are denoted with stars.

$$\Omega^{**} := J^{**} + \sum_{j=1}^{n(x)} V_j^{**} + V_i^{**}$$

$$\Omega^{*} := J^{*} + \sum_{j=1}^{n(x)} V_j^{*}$$

$$\Omega := J + \sum_{j=1}^{n(x)} V_j$$

Proceeding by backward induction, under **A-EN** the firm makes a take-it-or-leave-it offer to the unemployed worker, therefore

$$V_i^{**} = U$$

We now divide the proof in several steps. We start by proving that for all incumbent workers $j = 1...n(x)$, $V_j^{**} = V_j^{*}$. We then use **A-IN** to argue that $\Omega^{*} = \Omega$. Once these claims have been proven, we move on to proving **C-UE** (*UE* Hires) and the part of for threats from unemployment **C-RT** (*UE* Threats). Finally, we show that our microfoundations for the renegotiation game deliver **A-IN**.

**Claim 1:** For all incumbents workers $j = 1...n(x)$, we have $V_j^{**} = V_j^{*}$.

We proceed by backwards induction using our assumptions **A-EN** and **A-IN**. Immediately after (IR1) has taken place, only the following events can happen:

1. Hire/not-hire
   - Either the worker is hired from unemployment (H),
   - Or the worker is not hired from unemployment (NH)
2. Possible new round of internal negotiation (IR2). This possible second round of internal negotiation (now including the newly hired worker) leads to values $V_j^{**}$.

We focus on subgame perfect equilibria in this multi-stage game. Therefore, after (IR1), workers perfectly anticipate what the outcome of the hire/not-hire stage will be. That is, after (IR1), they know perfectly what hiring decision (H or NH) the firm will make. Now suppose that internal renegotiation (IR2) actually happens after the hire/not-hire decision, that is, that for some incumbent worker $j \in$

$\{1, ..., n(x)\}$, $V_j^{**} \neq V_j^*$. Note that the firm has no incentives to accept a change in the new worker's value to anything above $U$, so by **A-MC** her value does not change in the second round (IR2).

We construct the rest of the proof by contradiction. Consider for a contradiction an incumbent worker $j$ whose value changed in (IR2). Because of **A-MC**, her value can change only in the following cases:

- The firm has a credible threat to fire worker $j$, in which case $V_j^{**} < V_j^*$
- Worker $j$ has a credible threat to quit, in which case $V_j^{**} > V_j^*$

In addition, those credible threats can lead to a different outcome than in (IR1), and thus $V_j^{**} \neq V_j^*$, only if the threat on either side was not available in (IR1). If that same threat was available in the first round (IR1), then the outcome of the bargaining (IR1) would have been $V_j^{**}$.

Recall that both incumbent worker $j$ and the firm understand and anticipate which hire/not-hire decision the firm will make after the first round (IR1). They also understand and anticipate that, in case of hire, the value of the new worker will remain $U$ in the second round (IR2).

Therefore, the firm can *credibly threaten* to hire the new worker in the first round *if and only if* it actually hires her after the first round (IR1) is over. This implies that the firm can credibly threaten worker to fire $j$ in the second round (IR2), by **A-LC**, *if and only if* it could credibly threaten her with hiring the new worker *in the first round of internal renegotiation (IR1).* This in turn entails that any credible threat the firm can make in the second round (IR2) was already available in the first round.

On the worker side, quitting into unemployment is a credible threat when her value is below the value of unemployment. So this threat does not change between the first round (IR1) and the second round (IR2), because the equilibrium value to that worker will always be above the value of unemployment.

In sum, the set of credible threats both to the firm and to worker $j$ does not change between the initial round of internal renegotiation (IR1) and the post-hiring-decision round (IR2). This finally implies that the outcome of the initial round of internal renegotiation (IR1) for any incumbent $j$ remains unchanged in the second round (IR2), that is:

$$V_j^{**} = V_j^*$$

which proves **Claim 1**.

We can now move on to proving **C-UE**.

**Proof of C-UE.** Using the definitions of $\Omega^{**}$ and $\Omega$, we can write

$$\Omega^{**} - \Omega = \left[ J^{**} + \sum_{j=1}^{n(x)} V_j^{**} + V_i^{**} \right] - \left[ J + \sum_{j=1}^{n(x)} V_j \right]$$

Now using $V_i^{**} = U$, we obtain

$$\Omega^{**} - \Omega = \left[ J^{**} + \sum_{j=1}^{n(x)} V_j^{**} \right] - \left[ J + \sum_{j=1}^{n(x)} V_j \right] + U$$

Using **Claim 1:** $V_j^{**} = V_j^{*}$, and adding and subtracting $J^*$ we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] + \left[ J^* + \sum_{j=1}^{n(x)} V_j^{*} \right] - \left[ J + \sum_{j=1}^{n(x)} V_j \right] + U$$

Subsituting in the definition of $\Omega$ and of $\Omega^*$,

$$\Omega^{**} - \Omega = [J^{**} - J^*] + [\Omega^* - \Omega] + U$$

Finally recall that internal renegotiation is (1) individually rational, and (2) is a zero-sum game, according to **A-IN**. Thus, all incumbent workers remain in the coalition after internal renegotiation, and the joint value is unchanged: $\Omega^* = \Omega$. Using $\Omega^* = \Omega$

$$\Omega^{**} - \Omega = [J^{**} - J^*] + U$$

which can be re-written

$$J^{**} - J^* = [\Omega^{**} - \Omega] - U$$

Now under **A-LC**, the firm will only hire if its value after hiring is higher than its value after internal renegotiation: $J^{**} - J^* \geq 0$. This inequality requires

$$\Omega^{**} - \Omega \geq U$$
$$\boldsymbol{\Omega} \left( h_U \left( x \right) \right) - \Omega \left( x \right) \geq U$$

The firm does not hire when its value of hiring is below its value of renegotiation $J^{**} < J^{*}$. This inequality implies

$$\Omega^{**} - \Omega < U$$

When the firm does not hire, we obtain using again **A-IN** and $\Omega^{*} = \Omega$:

$$\Omega^{**} - \Omega^{*} < U$$

which finally implies

$$\mathbf{\Omega}\left(h_U\left(x\right)\right) - \mathbf{\Omega}\left(t_U(x)\right) < U$$

Now, we argue that conditional on not hiring, $\Omega^{**} = \Omega^{*} = \Omega$, where in this case $\Omega^{**}$ denotes the value of the coalition without hiring, and thus does not include the value of the unemployed worker. Just as before, this is a direct consequence from **A-IN** and that the internal renegotiation game is zero-sum.

Therefore:

$$\mathbf{\Omega}\left(t_U(x)\right) = \Omega(x)$$

We have therefore shown **C-UE** and part of **C-RT (*UE* Hires and *UE* Threats)**: An unemployed worker that meets $x$ is hired when$x \in \mathcal{Q}^U$, where

$$\mathcal{A} = \left\{x \middle| \mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) \geq U\right\}$$

and upon joining the firm, has value

$$\boldsymbol{V}\left(h_U\left(x,i\right)\right) = U.$$

and

$$\mathbf{\Omega}(t_U(x)) = \Omega(x).$$

### C.3.2 Proof of C-EE and C-RT (*EE* Hires, *EE* Threats and Retentions)

Consider firm $x$ that has met worker $i'$ at firm $x'$. We first seek to determine $\mathcal{Q}^E\left(x', i'\right)$. Under **A-IN** and **A-EN**, upon meeting an employed worker, internal negotiation may take place at the poaching firm $x$, and $x$ makes a take-it-or-leave-it offer. Internal negotiation may take place at $x'$ with all workers including $i'$.

Proceeding by backward induction, we again define intermediate values but here at $x'$, noting that

$q_E(x', i', x)$ gives the number of employees in $x'$ if the worker leaves:

$$\Omega = J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}$$

$$\Omega^* = J^* + \sum_{j=1}^{n(q_E(x',i',x))} V_j^* + V_{i'}^*$$

$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**}$$

Note, in the second line we are describing the values of the firm in renegotiation where $i'$ stays with the firm, so $V_{i'}^*$ is the outcome of internal negotiation. In the third line we consider the firm having lost the worker. Under **A-EN** the firm will respond to an offer $\overline{V}$ from $x$ with

$$V_{i'}^* = \overline{V}$$

The same result as in **Claim 1** from section C.3.1 obtains: under **A-EN** and **A-IN**, the values accepted by the incumbent workers *after the internal renegotiation* $\left(V_j^*\right)_j$ will be equal to the values they receive *after the external negotiation* $\left(V_j^{**}\right)_j$, that is

$$V_j^{**} = V_j^*$$

The argument are exactly the same.

Using these two results and the above definitions

$$\begin{aligned}
\Omega^{**} - \Omega &= \left[J^{**} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**}\right] - \left[J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}\right] \\
&= \left[J^{**} + J^* - J^* + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**} + V_{i'}^* - V_{i'}^*\right] - \left[J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}\right] \\
&= [J^{**} - J^*] + \left[J^* + \sum_{j=1}^{n(q_E(x',i',x))} V_j^* + V_{i'}^*\right] - \left[J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}\right] - V_{i'}^* \\
&= [J^{**} - J^*] + [\Omega^* - \Omega] - V_{i'}^* \\
&= [J^{**} - J^*] + [\Omega^* - \Omega] - \overline{V}
\end{aligned}$$

In this setup, **A-IN** again implies that any value lost to the firm must accrue to its workers, while any value lost to a worker must accrue either to the firm, or to another worker, which we earlier formulated as *"the joint value stays constant before and after an internal negotiation"*. Mathematically, this statement translates into

$$\Omega^* = \Omega$$

Subsituting into the equation that we obtained above $\Omega^{**} - \Omega = [J^{**} - J^*] + [\Omega^* - \Omega] - \overline{V}$, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] - \overline{V}$$

Now under **A-LC**, the firm $x'$ will only try to keep the worker if $J^* > J^{**}$, which requires

$$\Omega - \Omega^{**} \leq \overline{V}$$
$$\mathbf{\Omega}\left(r(x', i', x\right) - \mathbf{\Omega}\left(q_E\left(x', i', x\right)\right) \leq \overline{V}$$

This determined the maximum value that $x'$ can offer to the worker to retain them. Knowing that firm $x'$ can counter at most with $\overline{V} = \mathbf{\Omega}\left(r(x', i', x\right) - \mathbf{\Omega}\left(q_E\left(x', i', x\right)\right)$, then will firm $x$ successfully poach the worker?

First, note that the bargaining protocol implies that $x$ firm will offer $\overline{V}$ if it is making an offer, since it need not offer more. For firm $x$ the argument may proceed identically to the case of unemployment, simply replacing $U$ with $\overline{V}$. The result is that the firm will hire only if

$$\mathbf{\Omega}\left(h_E\left(x', i', x\right)\right) - \Omega\left(x\right) \geq \overline{V}$$

or

$$\mathbf{\Omega}\left(h_E\left(x', i', x\right)\right) - \Omega\left(x\right) \geq \mathbf{\Omega}\left(r(x', i', x)\right) - \mathbf{\Omega}\left(q_E\left(x', i', x\right)\right)$$

Finally, when firm $x$ does not hire, the same argument as in **Claim 32** in Section C.3.1 applies: $\Omega^{**} = \Omega^* = \Omega$. This observation implies

$$\mathbf{\Omega}(t_E(x', i', x)) = \Omega(x)$$

Similarly, the same argument as in **Claim 2** implies that when firm $x'$ does not lose its worker, $\Omega^{**} = \Omega^* = \Omega$, thereby implying

$$\mathbf{\Omega}(r(x', i', x)) = \Omega(x')$$

The combination of these conditions deliver **C-UE** and part of **C-RT** (*EE* Hires, *EE* Threats and Retention):

1. The quit set of an employed worker is determined by

$$\mathcal{Q}^E\left(x',i'\right)=\left\{x\middle|\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right)-\Omega\left(x\right)\geq\Omega\left(x'\right)-\boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)\right\}$$

2. The worker's value of being hired from employment from firm $x'$ is

$$\boldsymbol{V}(h_E(x,x',i'))=\Omega\left(x'\right)-\boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)$$

3. Worker $i'$s value of being retained at $x'$ after meeting $x$ is[4]

$$\boldsymbol{V}(r(x',i',x),i')=\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right)-\Omega\left(x\right)$$

4. The joint value of the potential poaching firm $x$ when the worker is not hired does not change:

$$\boldsymbol{\Omega}(t_E(x',i',x))=\Omega(x)$$

5. The joint value of the potential poached firm $x'$ does not change when the worker stays:

$$\boldsymbol{\Omega}(r(x',i',x))=\Omega(x')$$

#### C.3.3 Proof of C-EU ($EU$ Quits and layoffs)

We first show that

$$\begin{aligned}\mathcal{L}=\Big\{(x,i)\Big|&\Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=1\right]\right)\circ\left(1-q_U\left(x\right)\right)\right),i\right)+U\\&>\Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=0\right]\right)\circ\left(1-q_U\left(x\right)\right)\right),i\right)\Big\}\end{aligned}$$

from the firm side, then that

$$\begin{aligned}\mathcal{Q}^U=\Big\{(x,i)\Big|&\Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=1\right]\right)\right),i\right)+U\\&>\Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=0\right]\right)\right),i\right)\Big\}\end{aligned}$$

[4] Because offers are made at no cost, both firms always make an offer, even when they know that they cannot retain/hire the worker in equilibrium. This is exactly the same as in Postel-Vinay Robin (2002).

on the worker side.

**Part 1: Firm side** Consider a firm $x$ who is considering laying off worker $i$ for whom $q_{U,i}(x) = 0$. As above, we start with definitions, noting that $n(s(x, (1 - [\ell(x); \ell_i(x) = 1]) \circ (1 - q_U(x))))$ is the number of workers if $i$ is laid off.

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i$$
$$\Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*$$
$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}$$

Note that in the first line the coalition has still worker $i$ in it. In the second line, the firm and the worker $i$ have negotiated (and internal negotiation has determined $V_i^*$ which is what $i$ will get if they stay in the firm). In the third line, the worker has been fired and another round of negotiation has occurred among incumbents.

The same result as in **Claim 1** from section C.3.1 obtains: under **A-BP**, the values accepted by the incumbent workers *after the internal renegotiation* $\left(V_j^*\right)$ will be equal to the values they receive *after the external negotiation* $\left(V_j^{**}\right)$, that is $V_j^{**} = V_j^*$.

Using this result and the above definitions

$$\begin{aligned}
\Omega^{**} - \Omega &= \left[J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] \\
&= \left[J^{**} - J^* + J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^* - V_i^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] \\
&= [J^{**} - J^*] + \left[J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] - V_i^* \\
&= [J^{**} - J^*] + [\Omega^* - \Omega] - V_i^*
\end{aligned}$$

Using again **A-IN** to conclude that $\Omega^* = \Omega$, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] - V_i^*$$

Now under **A-LC**, the firm $x$ will only layoff the worker if $J^{**} > J^{*}$, which requires

$$\Omega - \Omega^{**} < V_i^*$$

As long as $V_i^* > U$ the worker would be willing to transfer value to the firm to avoid being laid off, implying

$$\Omega - \Omega^{**} < U.$$

which we can re-write

$$\Omega\left(s\left(x, \left(1 - \left[\ell\left(x\right); \ell_i\left(x\right) = 1\right]\right) \circ \left(1 - q_U\left(x\right)\right)\right), i\right) + U > \Omega\left(s\left(x, \left(1 - \left[\ell\left(x\right); \ell_i\left(x\right) = 0\right]\right) \circ \left(1 - q_U\left(x\right)\right)\right), i\right)$$

where the LHS is $\Omega^{**} + U$ (under the layoff) and the RHS is $\Omega$. This concludes the proof for the firm side.

**Part 2: Worker side** Consider worker $i$ in firm $x$ who is considering quitting to unemployment for whom $\ell_i\left(x\right) = 0$. As above, we start with definitions, noting that $n\left(s\left(x, \left(1 - \ell\left(x\right)\right) \circ \left(1 - \left[q_{U,-i}\left(x\right); q_{U,i}\left(x\right) = 1\right]\right)\right)\right)$ is the number of workers if $i$ quits. As before,

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i$$

$$\Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*$$

$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}$$

The same result as in **Claim 1** from section C.3.1 obtains $V_j^{**} = V_j^*$.

Using this result and the above definitions

$$\begin{aligned}
\Omega^{**} - \Omega &= \left[ J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**} \right] - \left[ J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i \right] \\
&= \left[ J^{**} + J^{*} - J^{*} + \sum_{j=1}^{n(s(\cdot))} V_j^{*} + V_i^{*} - V_i^{*} \right] - \left[ J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i \right] \\
&= [J^{**} - J^{*}] + \left[ J^{*} + \sum_{j=1}^{n(s(\cdot))} V_j^{*} + V_i^{*} \right] - \left[ J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i \right] - V_i^{*} \\
&= [J^{**} - J^{*}] + [\Omega^{*} - \Omega] - V_i^{*}
\end{aligned}$$

Again, $\Omega^{*} = \Omega$ from **A-IN**, so that

$$\Omega^{**} - \Omega = [J^{**} - J^{*}] - V_i^{*}$$

Now under **A-LC**, worker $i$ will quit into unemployment iff $V_i^{*} < U$, which requires

$$J^{**} - J^{*} + [\Omega - \Omega^{**}] < U$$

As long as $J^{**} < J^{*}$, the firm is willing to transfer value to worker $i$ to retain her. Therefore, worker $i$ quits into unemployment iff the previous inequality holds at $J^{**} = J^{*}$, i.e.

$$\Omega - \Omega^{**} < U$$

Therefore, the worker quits iff

$$\begin{aligned}
&\Omega\left(s\left(x, \left(1 - \ell\left(x\right)\right) \circ \left(1 - \left[q_{U,-i}\left(x\right); q_{U,i}\left(x\right) = 1\right]\right)\right), i\right) + U \\
&> \Omega\left(s\left(x, \left(1 - \ell\left(x\right)\right) \circ \left(1 - \left[q_{U,-i}\left(x\right); q_{U,i}\left(x\right) = 0\right]\right)\right), i\right)
\end{aligned}$$

which concludes the proof of the worker side. This delivers **C-EU**.

### C.3.4 Proof of C-X (Exit)

Consider a firm $x$ who contemplates exit after all endogenous quits and layoffs, thus when its employment is $n(s(x,\kappa(x)))$. As before we define values conditional on exiting:

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j$$

$$\Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^*$$

$$\Omega^{**} = J^{**} + 0$$

Notice that the joint value after exit is simply the value of the firm, since all other workers have left because of exit. We can compute:

$$\begin{aligned}
\Omega^{**} - \Omega &= J^{**} - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j\right] \\
(\text{add and subtract } J^*) &= [J^{**} - J^*] + J^* - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j\right] \\
(\text{add and subtract } \sum_{j=1}^{n(s(\cdot))} V_j^*) &= [J^{**} - J^*] + \left[J^* + \sum_{j=1}^{n(s(\cdot))} V_j^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j\right] - \sum_{j=1}^{n(s(\cdot))} V_j^* \\
(\text{definition of } \Omega, \Omega^*) &= [J^{**} - J^*] + [\Omega^* - \Omega] - \sum_{j=1}^{n(s(\cdot))} V_j^*
\end{aligned}$$

Again, $\Omega^* = \Omega$ from **A-IN**, so that

$$\Omega^{**} - \Omega = [J^{**} - J^*] - \sum_{j=1}^{n(s(\cdot))} V_j^*$$

The firm exits iff $J^{**} \geq J^*$, that is, $\vartheta \geq J^*$. This is equivalent to

$$\Omega^{**} - \Omega \geq - \sum_{j=1}^{n(s(\cdot))} V_j^*$$

Using again that $\Omega^{**} = J^{**} = \vartheta$, the firm exits iff

$$\vartheta + \sum_{j=1}^{n(s(\cdot))} V_j^* \geq \Omega$$

Since any worker is better off under $V_i^* \geq U$ than unemployed, all workers are willing to take a value cut down to $U$ if $\vartheta \geq \Omega - \sum_{j=1}^{n(s(\cdot))} V_j^*$ because then the firm can credibly exit.

This implies that the firm exits if and only if

$$\vartheta - \Omega\left(s\left(x, \kappa\left(x\right)\right)\right) + n\left(s\left(x, \kappa\left(x\right)\right)\right) U \geq 0$$

This proves **C-X (Exit)**: the set of $x$ such that the firm exits is given by

$$\mathcal{E} = \left\{ x \middle| \vartheta + n\left(s\left(x, \kappa\left(x\right)\right)\right) \cdot U \geq \Omega\left(s\left(x, \kappa\left(x\right)\right)\right) \right\}$$

### C.3.5 Proof of C-V (Vacancies)

We split the proof in two steps. First, we show that workers are collectively willing to transfer value to the firm in exchange for the joint value-maximizing vacancy policy function. Second, we show that a single worker can create a system of transfers that achieves the same outcome. These transfers are equivalent to wage renegotiation, which explains why we have subsumed them in the wage function $w(x, i)$ in the equations above. Similarly to wages, these transfers drop out from the expression for the joint value.

**Part 1: Collective transfers** In this step, we show that workers are collectively better off transferring value to the firm in exchange of the firm posting the joint value-maximizing amount of vacancies.

The vacancy posting decision $v^J$ that maximizes firm value is:

$$\frac{c_v\left(v^J\left(x\right), n\left(x\right)\right)}{q} = \phi\left[\boldsymbol{J}\left(h_U\left(x\right)\right) - J\left(x\right)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} + (1-\phi) \int_{x \in \mathcal{Q}^E(x', i')} \left[\boldsymbol{J}\left(h_E\left(x', i', x\right)\right) - J\left(x\right)\right] dH_n\left(x', i'\right).$$

Similarly, define $v^\Omega$ be the policy that maximizes the value of the coalition, and $v^{\overline{V}}$ be the policy that maximizes the value of all the employees. Let $\Omega^\gamma, J^\gamma, \overline{V}^\gamma$ be the value of the coalition, firm and all workers under the $v^\gamma$, for $\gamma \in \left\{\Omega, J, \overline{V}\right\}$. We now prove our claim in several steps.

**Part 1-(a) Collective value gains.** The policy $v^\Omega$ will lead to $\overline{V}^\Omega \geq \overline{V}^J + [J^J - J^\Omega]$ where $J^J - J^\Omega \geq 0$.

*Proof:* By construction $\Omega^\Omega$ is greater than $\Omega^J$: $\Omega^\Omega \geq \Omega^J$. By definition: $\Omega^\Omega = J^\Omega + \overline{V}^\Omega$, and $\Omega^J = J^J + \overline{V}^J$. Use those definitions to obtain inequality $J^\Omega + \overline{V}^\Omega \geq J^J + \overline{V}^J$, which can be re-arranged into $\overline{V}^\Omega - \overline{V}^J \geq J^J - J^\Omega$. Since $J^J$ is the value under the optimal policy for $J$, then $J^J \geq J^\Omega$. The above then implies that

$$\overline{V}^{\Omega} - \overline{V}^{J} \geq J^{J} - J^{\Omega} \geq 0$$

This implies that workers would be prepared to transfer $T = J^{J} - J^{\Omega} \geq 0$ to the firm in order for the firm to pursue policy $v^{\Omega}$ instead of $v^{J}$. This concludes the proof of **Part 1-(a).**

**Part 1-(b) Infeasibility of $\overline{V}^{\overline{V}}$.** There does not exist an incentive-compatible transfer from workers to firm that will lead to $\overline{V}^{\overline{V}}$.

*Proof:* Suppose workers consider transferring even more to induce the firm to follow policy $v^{\overline{V}}$ that maximizes their value. By construction $\Omega^{\Omega} \geq \Omega^{\overline{V}}$. Using definitions for each of these, then $J^{\Omega} + \overline{V}^{\Omega} \geq J^{\overline{V}} + \overline{V}^{\overline{V}}$. Rearranging this: $J^{\Omega} - J^{\overline{V}} \geq \overline{V}^{\overline{V}} - \overline{V}^{\Omega}$. Since $\overline{V}^{\overline{V}}$ is the value under the optimal policy for $\overline{V}$, then $\overline{V}^{\overline{V}} \geq \overline{V}^{\Omega}$. The above then implies that

$$J^{\Omega} - J^{\overline{V}} \geq \overline{V}^{\overline{V}} - \overline{V}^{\Omega} \geq 0$$

Taking $v^{\Omega}$ as a baseline, the above implies that a change to $v^{\overline{V}}$ causes a loss of $J^{\Omega} - J^{\overline{V}}$ to the firm, which is more than the gain of $\overline{V}^{\overline{V}} - \overline{V}^{\Omega}$ to the workers. This implies that workers could transfer all of their gains under $v^{\overline{V}}$ to the firm, but the firm would still not choose $v^{\overline{V}}$ over $v^{\Omega}$. This concludes the proof of **Part 1-(b).**

**Part 1-(c) Optimality of $\overline{V}^{\Omega}$.** There does not exist an incentive-compatible transfer from workers to firm that will lead to $\overline{V}^{*} \in \left(\overline{V}^{\Omega}, \overline{V}^{\overline{V}}\right)$.

*Proof:* Call such a policy $v^{\overline{V}*}$. Then: $\Omega^{\Omega} \geq \Omega^{\overline{V}*}$, and by definitions

$$\begin{aligned} J^{\Omega} + \overline{V}^{\Omega} &\geq J^{\overline{V}^{*}} + \overline{V}^{\overline{V}^{*}} \\ J^{\Omega} - J^{\overline{V}^{*}} &\geq \overline{V}^{\overline{V}^{*}} - \overline{V}^{\Omega} \end{aligned}$$

Since by definition $\overline{V}^{*} \in \left(\overline{V}^{\Omega}, \overline{V}^{\overline{V}}\right)$, then $\overline{V}^{\overline{V}^{*}} - \overline{V}^{\Omega} \geq 0$. Therefore

$$J^{\Omega} - J^{\overline{V}^{*}} \geq \overline{V}^{\overline{V}^{*}} - \overline{V}^{\Omega} \geq 0$$

Taking $v^{\Omega}$ as a baseline, the above implies that a change to $v^{\overline{V}^{*}}$ causes a loss of $J^{\Omega} - J^{\overline{V}^{*}}$ to the firm, which is more than the gain of $\overline{V}^{\overline{V}^*} - \overline{V}^{\Omega}$ to the workers. This concludes the proof of **Part 1-(c).**

**Part 1-(d) Conclusion.** In summary, it is optimal for workers to transfer exactly $T = J^J - J^{\Omega}$ to the firm, in order for the firm to pursue $v^{\Omega}$ instead of $v^J$. Further transfers to the firm would be required to have the firm pursue a better policy for workers, but this is exceedingly costly to the firm and the workers are unwilling to make a transfer to cover these costs. This concludes the proof of **Step 1: Collective transfers.**

**Part 2: Individual transfers** In this step, we show that a single, randomly drawn worker can construct a system of transfers that induces the firm to post $v^{\Omega}$ instead of $v^J$, while leaving all agents better off.

Within $dt$, consider the single, randomly drawn worker $j_0$. Consider the following system of transfers. Worker $j_0$ makes a transfer $J^J - J^{\Omega}$ to the firm, in exchange of what (i) the firm posts $v^{\Omega}$ instead of $v^J$, and (ii) the worker gets a wage increase that gives her all the differential surplus $\overline{V}^{\Omega} - \overline{V}^J$.

Following the same steps as in **Part 1: Collective transfers**, the firm gets $J^{\Omega} + [J^J - J^{\Omega}] = J^J$ and is hence indifferent. Similarly, workers $j \neq j_0$ do not get any value change, and are thus indifferent Finally, worker $j_0$ gets a value increase of

$$[\overline{V}^{\Omega} - \overline{V}^J] - [J^J - J^{\Omega}] \geq 0$$

where the inequality similarly follows from **Part 1: Collective transfers**. This concludes the proof of **Part 2: Individual transfers**.

**Conclusion.** The previous arguments show that a single worker has an incentive to and can induce the firm to post $v^{\Omega}$. Notice also that the same argument holds starting from any vacancy policy function $\widetilde{v} \neq v^J$ together with a value of the firm $\widetilde{J}$. Thus, even if some worker induces the firm to post a different vacancy policy function which is not $v^{\Omega}$ any other worker has an incentive to induce the firm to post $v^{\Omega}$. Therefore, in equilibrium, the firm posts $v^{\Omega}$, which concludes the proof of **C-V**.

## C.4 Applying Conditions (C)

Having established that **Assumption (A)** can be used to prove **Conditions (C)**, we now apply conditions **(C)** to the Bellman equation for the joint value. The goal of this section is to show that for $x \in \mathcal{E}^c$ the complement of the exit set, we can considerably simplify the recursion for the joint value:

$$
\begin{aligned}
\rho\Omega\left(x\right) &= y\left(z(x), n\left(x\right)\right) - c\left(v\left(x\right), n\left(x\right), z(x)\right) \\
\text{Destructions} \quad & -\delta \sum_{i=1}^{n(x)} \left[\Omega\left(x\right) - \mathbf{\Omega}\left(d(x,i)\right) - U\right] \\
UE \text{ Hires} \quad & +qv\left(x\right)\phi\left[\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) - U\right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
EE \text{ Hires} \quad & +qv\left(x\right)\left(1-\phi\right)\int_{x\in\mathcal{Q}^E(x',i')} \left[\left[\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right)\right] - \left[\Omega\left(x'\right) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)\right]\right] dH_n\left(x',i'\right) \\
\text{Shocks} \quad & +\Gamma[\mathbf{\Omega}, \Omega]
\end{aligned}
$$

with the sets

$$
\begin{aligned}
\mathcal{Q}^U &= \Big\{(x,i)\Big|\Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right); q_{U,i}\left(x\right)=1\right]\right)\right), i\right) + U \\
&\qquad > \Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right); q_{U,i}\left(x\right)=0\right]\right)\right), i\right)\Big\} \\
\mathcal{L} &= \Big\{(x,i)\Big|\Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=1\right]\right)\circ\left(1-q_U\left(x\right)\right)\right), i\right) + U \\
&\qquad > \Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=0\right]\right)\circ\left(1-q_U\left(x\right)\right)\right), i\right)\Big\} \\
\mathcal{E} &= \Big\{x\Big|\vartheta + n\left(s(x,\kappa(x))\right)\cdot U \geq \Omega(s(x,\kappa(x)))\Big\} \\
\mathcal{A} &= \Big\{x\Big|\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) \geq U\Big\} \\
\mathcal{Q}^E\left(x',i'\right) &= \left\{x\middle|\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right) \geq \Omega\left(x'\right) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)\right\}
\end{aligned}
$$

and—as per **(C-V)—**the vacancy policy $v\left(x\right)$ is given by the solution to the following:

$$
\begin{aligned}
\frac{c_v\left(v\left(x\right), n\left(x\right)\right)}{q} &= \phi\left[\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right)\right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
&+ \left(1-\phi\right)\int_{x\in\mathcal{Q}^E(x',i')} \left[\left[\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right)\right] - \left[\Omega\left(x'\right) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)\right]\right] dH_n\left(x',i'\right)
\end{aligned}
$$

In continuous time, the exit decision is captured by $x \in \mathcal{E}$. The Bellman equation above holds exactly for $x \in \mathcal{E}^c$. Exit is accounted for in the "bold" continuation values, which all include the possible exit decision should the firm's state fall into $\mathcal{E}$ after an event.

We first proceed one term at the time, working through (B.4.1) exogenous destructions, (B.4.2) retentions, (B.4.3) *EE* (poached) quits, (B.4.4) *UE* hires, (B.4.5) *UE* threats, (B.4.6) *EE* (poached) hires, and (B.4.7) *EE* threats.

### C.4.1 Exogenous destructions

$$\begin{aligned}\text{Destructions} &= \sum_{i=1}^{n(x)} \delta \left[ \boldsymbol{J}\left(d(x,i)\right) + \sum_{j=1}^{n(d(x,i))} \boldsymbol{V}\left(d(x,i),j\right) + U - \Omega\left(x\right) \right] \\ &= \sum_{i=1}^{n(x)} \delta \left[ \boldsymbol{\Omega}\left(d(x,i)\right) + U - \Omega\left(x\right) \right]\end{aligned}$$

where we simply have used the definition $\boldsymbol{\Omega}\left(d(x,i)\right) := \boldsymbol{J}\left(d(x,i)\right) + \sum_{j=1}^{n(d(x,i))} \boldsymbol{V}\left(d(x,i),j\right)$.

### C.4.2 Retentions

$$\begin{aligned}\text{Retentions} &= \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[ \boldsymbol{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right) - \Omega\left(x\right) \right] dH_v\left(x'\right) \\ &= \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[ \boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) - \Omega\left(x\right) \right] dH_v\left(x'\right)\end{aligned}$$

where we simply have used the definition $\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) = \boldsymbol{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right)$. Now using the result in **C-RT** that

$$\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) = \Omega(x')$$

we obtain that

$$\text{Retentions} = 0$$

### C.4.3 *EE* Quits

$$EE\ \text{Quits} = \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[ \boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) + \boldsymbol{V}\left(q_E\left(x,i,x'\right),i\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - \Omega\left(x\right) \right] dH_v\left(x'\right)$$

Now by definition

$$\begin{aligned}\boldsymbol{\Omega}\left(q_E\left(x,i,x'\right)\right) &= \boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) + \sum_{j=1}^{n(q_E(x,i,x'))} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) \\ &= \boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right)\end{aligned}$$

Using this last equality in the term in square brackets

$$EE \text{ Quits} = \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[ \Omega \left( q_E \left( x, i, x' \right) \right) - \Omega \left( x \right) + \boldsymbol{V} \left( q_E \left( x, i, x' \right), i \right) \right] dH_v \left( x' \right)$$

Using **C-EE**, the value going to the poached worker is $\boldsymbol{V} \left( q_E \left( x, i, x' \right) \right) = \Omega \left( x \right) - \Omega \left( q_E \left( x, i, x' \right) \right)$. Substituting this into the last equation, we observe that the term in the square brackets is zero, and so

$$EE \text{ Quits} = 0$$

### C.4.4 *UE* Hires

$$UE \text{ Hires} = qv\left(x\right)\phi \left[ \boldsymbol{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_U\left(x\right), i\right) - \Omega\left(x\right) \right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}$$

Now by definition

$$\begin{aligned} \boldsymbol{\Omega}\left(h_U\left(x\right)\right) &= \boldsymbol{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(h_U(x))} \boldsymbol{V}\left(h_U\left(x\right), i\right) \\ &= \boldsymbol{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_U\left(x\right), i\right) + \boldsymbol{V}\left(h_U\left(x\right), i\right) \end{aligned}$$

and so, re-arranging,

$$\boldsymbol{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_U\left(x\right), i\right) = \boldsymbol{\Omega}\left(h_U\left(x\right)\right) - \boldsymbol{V}\left(h_U\left(x\right), i\right)$$

Substituting this last equation into the term in the square brackets of the first equation,

$$UE \text{ Hires} = qv\left(x\right)\phi \left[ \boldsymbol{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) - \boldsymbol{V}\left(h_U\left(x\right), i\right) \right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}$$

Following **C-UE**, the value going to the hired worker is $\boldsymbol{V}\left(h_U\left(x\right), i\right) = U$. Substituting in:

$$UE \text{ Hires} = qv\left(x\right)\phi \left[ \boldsymbol{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) - U \right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}$$

### C.4.5 *UE* Threats

$$UE \text{ Threats} \quad = \quad qv\left(x\right)\phi \left[ \boldsymbol{J}\left(t_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_U\left(x\right), i\right) - \Omega\left(x\right) \right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}}$$

Using the definition of $\mathbf{\Omega}(t_U(x))$, we can re-write this term as

$$UE \text{ Threats} = qv(x)\,\phi\left[\mathbf{\Omega}\left(t_U(x)\right) - \Omega(x)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}}$$

Now using our result in condition **C-UE** that $\mathbf{\Omega}\left(t_U(x)\right) = \Omega(x)$, we can conclude that

$$UE \text{ Threats} = 0$$

### C.4.6 *EE* Hires

$$EE \text{ Hires} = qv(x)(1-\phi)\int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) - \Omega(x)\right] dH_n\left(x',i'\right)$$

Now by definition

$$\begin{aligned}
\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) &= \boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(h_E(x',i',x))} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) \\
&= \left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)\right] + \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)
\end{aligned}$$

which can be re-arranged into

$$\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) = \mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)$$

Using this in the term in the square brackets

$$EE \text{ Hires} = qv(x)(1-\phi)\int_{x \in \mathcal{Q}^E(x',i')} \left[\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega(x) - \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)\right] dH_n\left(x',i'\right)$$

Under **C-EE**, the value going to the hired worker is $\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) = \Omega\left(x'\right) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)$. Substituting this in:

$$EE \text{ Hires} = qv(x)(1-\phi)\int_{x \in \mathcal{Q}^E(x',i')} \left[\left[\mathbf{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega(x)\right] - \left[\Omega\left(x'\right) - \mathbf{\Omega}\left(q_E\left(x',i',x\right)\right)\right]\right] dH_n\left(x',i'\right)$$

### C.4.7 *EE* Threats

$$EE \text{ Threats} = qv(x)(1-\phi)\int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(t_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - J(x) - \overline{V}(x)\right] dH_n\left(x',i'\right)$$

Using the definition of $\mathbf{\Omega}(t_E(x',i',x))$, we obtain

$$EE \text{ Threats} = qv(x)(1-\phi)\int_{x\notin \mathcal{Q}^E(x',i')} \left[\mathbf{\Omega}\left(t_E\left(x',i',x\right)\right) - \Omega(x)\right] dH_n\left(x',i'\right)$$

Now using the result in condition C-RT that $\mathbf{\Omega}\left(t_E\left(x',i',x\right)\right) = \Omega(x)$, we obtain that

$$EE \text{ Threats} = 0$$

## C.5 Reducing the state space

Now that we obtained the simplified recursion, we are in a position to argue that the only payoff-relevant states are $(z,n)$, and that the details of the within-firm contractual structure do not affect allocations. The goal of this section is to show that we can express the joint values pre- and post- separation and exit decisions as follows. First, the exit and separation decisions are characterized by

$$\begin{aligned} \mathbf{\Omega}(z,n) &= \mathbb{I}_{\{(z,n)\in\mathcal{E}\}}\left\{\vartheta + nU\right\} + \mathbb{I}_{\{(z,n)\in\mathcal{Q}^U\}}\left\{\mathbf{\Omega}(z,n-1)+U\right\} + \mathbb{I}_{\{(z,n)\notin\mathcal{Q}^U\cup\mathcal{E}\}}\Omega(z,n), \quad (5)\\ \text{where}\quad \mathcal{E} &= \left\{n,z\middle|\vartheta + nU > \Omega(z,n)\right\},\\ \mathcal{Q}^U &= \left\{z,n\middle|\mathbf{\Omega}(z,n-1)+U > \Omega(z,n)\right\}. \end{aligned}$$

The first expression is the value of exit. A firm that does not exit, chooses whether to separate with a worker or not. If separating with a worker, the firm re-enters (5) with $\mathbf{\Omega}(z,n-1)$, having dispatched with a worker with value $U$, and again choosing whether to exit, fire another worker, or continue. Iterating on this procedure delivers

$$\mathbf{\Omega}(z,n) = \max\left\{\vartheta + nU\,,\ \max_{s\in[0,\ldots,n]}\Omega(z,n-s)+sU\right\}. \quad (6)$$

Second, the post-exit/separation decision joint value is given by the Bellman equation

$$
\begin{aligned}
\rho\Omega\left(z,n\right) &= \max_{v\geq 0} y\left(z,n\right) - c\left(v,n,z\right) \\
\text{Destruction} \quad &+ \quad \delta n \Big\{ \left(\boldsymbol{\Omega}\left(z,n-1\right) + U\right) - \Omega\left(z,n\right) \Big\} \\
UE \text{ Hire} \quad &+ \quad \phi q(\theta) v \cdot \mathbb{I}_{\{(z,n)\in\mathcal{A}\}} \cdot \Big\{ \boldsymbol{\Omega}\left(z,n+1\right) - \left(\Omega\left(z,n\right) + U\right) \Big\} \\
EE \text{ Hire} \quad &+ \quad \left(1-\phi\right) q(\theta) v \int_{(z,n)\in\mathcal{Q}^E(z',n')} \Big\{ \left[\boldsymbol{\Omega}\left(z,n+1\right) - \Omega\left(z,n\right)\right] - \left[\Omega\left(z',n'\right) - \boldsymbol{\Omega}\left(z',n'-1\right)\right] \Big\} dH_n\left(z',n'\right) \\
\text{Shock} \quad &+ \quad \Gamma_z\left[\boldsymbol{\Omega},\Omega\right]\left(z,n\right), \\
\text{where} \quad \mathcal{A} &= \Big\{ z,n \Big| \boldsymbol{\Omega}\left(z,n+1\right) \geq \Omega\left(z,n\right) + U \Big\}, \\
\mathcal{Q}^E\left(z',n'\right) &= \Big\{ z,n \Big| \boldsymbol{\Omega}\left(z,n+1\right) - \Omega\left(z,n\right) \geq \Omega\left(z',n'\right) - \boldsymbol{\Omega}\left(z',n'-1\right) \Big\}.
\end{aligned}
$$

Finally, firms enter if and only if

$$\int \boldsymbol{\Omega}\left(z,0\right) d\Pi_0(z) \geq c_e. \tag{7}$$

This condition pins down the entry rate per unit of time.[5]

We proceed in three steps. First, we isolate $(z,n)$ in the state vector $x$ by writing $x = (z,n,\chi)$ where $\chi$ collects all other terms in $x$. Second, we introduce functions that update $\chi$ following events to the firm and worker. Third, we argue that $\chi$ is a redundant state. This delivers the final Bellman equation for the joint value function for the discrete workforce model, equation (7).

[5]Recall that $J_0 = -c_e + \int \boldsymbol{J}(x_0) d\Pi(z_0)$. Given $\Omega(z_0,0) = J(z_0,0)$, we have $J_0 = -c_e + \int \boldsymbol{\Omega}(z_0,0) d\Pi(z_0)$. Free-entry implies $J_0 = 0$, which delivers (7).

### C.5.1 Isolate $(z, n)$ in the state vector

It is immediate that $x$ should contain at least the pair $(z, n)$. Call everything else $\chi$. Then we express $x = (z, n, \chi)$. Making this substitution into the above conditions:

$$\begin{aligned}
\rho\Omega(z,n,\chi) \quad &= \quad y(z,n) - c(v(z,n,\chi), n) \\
\text{Destructions} \quad & \quad -\delta \sum_{i=1}^{n(x)} [\Omega(z,n,\chi) - \mathbf{\Omega}(d(z,n,\chi,i)) - U] \\
UE \text{ Hires} \quad & \quad +qv(z,n,\chi)\,\phi\,[\mathbf{\Omega}(h_U(z,n,\chi)) - \Omega(z,n,\chi) - U] \cdot \mathbb{I}_{\{(z,n,\chi)\in\mathcal{A}\}} \\
EE \text{ Hires} \quad & \quad +qv(z,n,\chi)(1-\phi)\int_{(z,n,\chi)\in\mathcal{Q}^E(n',z',\chi',i')} \Big[ \left[\mathbf{\Omega}\left(h_E\left(z',n',\chi',i',z,n,\chi\right)\right) - \Omega(z,n,\chi)\right] \\
& \qquad - \left[\Omega\left(z',n',\chi',i'\right) - \mathbf{\Omega}\left(q_E\left(n',z',\chi',i',z,n,\chi\right)\right)\right] \Big] \\
& \qquad \cdot dH_n\left(z',n',\chi',i'\right) \\
\text{Shocks} \quad & \quad +\Gamma_z[\mathbf{\Omega}, \Omega](z,n,\chi)
\end{aligned}$$

with sets

$$\begin{aligned}
\mathcal{Q}^U &= \Big\{ (z,n,\chi,i) \Big| \Omega(s(z,n,\chi,(1-\ell(z,n,\chi)) \circ (1-[q_{U,-i}(z,n,\chi); q_{U,i}(z,n,\chi)=1])), i) + U \\
&\qquad > \Omega(s(z,n,\chi,(1-\ell(z,n,\chi)) \circ (1-[q_{U,-i}(z,n,\chi); q_{U,i}(z,n,\chi)=0])), i) \Big\} \\
\mathcal{L} &= \Big\{ (z,n,\chi,i) \Big| \Omega(s(z,n,\chi,(1-[\ell(z,n,\chi); \ell_i(z,n,\chi)=1]) \circ (1-q_U(z,n,\chi))), i) + U \\
&\qquad > \Omega(s(z,n,\chi,(1-[\ell(z,n,\chi); \ell_i(z,n,\chi)=0]) \circ (1-q_U(z,n,\chi))), i) \Big\} \\
\mathcal{E} &= \Big\{ z,n,\chi \Big| \vartheta + n(s(z,n,\chi,\kappa(z,n,\chi))) \cdot U \geq \Omega(s(z,n,\chi,\kappa(z,n,\chi))) \Big\} \\
\mathcal{A} &= \Big\{ z,n,\chi \Big| \Omega(h_U(z,n,\chi)) - \Omega(z,n,\chi) \geq U \Big\} \\
\mathcal{Q}^E\left(z',n',\chi',i'\right) &= \Big\{ z,n,\chi \Big| \mathbf{\Omega}\left(h_E\left(z',n',\chi',i',z,n,\chi\right)\right) - \Omega(z,n,\chi) \geq \Omega\left(n',z',\chi',i'\right) - \mathbf{\Omega}\left(q_E\left(z',n',\chi',i',z,n,\chi\right)\right) \Big\}
\end{aligned}$$

and vacancy posting

$$
\begin{aligned}
\frac{c_v\left(v\left(z,n,\chi\right),z,n\right)}{q} &= \phi\left[\boldsymbol{\Omega}\left(h_U\left(z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)\right]\cdot\mathbb{I}_{\{(z,n,\chi)\in\mathcal{A}\}} \\
&+(1-\phi)\int_{(z,n,\chi)\in\mathcal{Q}^E(z',n',\chi',i')}\Big[\left[\boldsymbol{\Omega}\left(h_E\left(z',n',\chi',i',z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)\right] \\
&\qquad -\left[\Omega\left(z',n',\chi',i'\right)-\boldsymbol{\Omega}\left(q_E\left(n',z',\chi',i',z,n,\chi\right)\right)\right]\Big] \\
&\qquad \cdot\, dH_n\left(z',n',\chi',i'\right)
\end{aligned}
$$

Finally, note that the contribution of shocks writes explicitly

$$
\Gamma_z[\boldsymbol{\Omega},\Omega]=\lim_{dt\to 0}E_t\left[\frac{\boldsymbol{\Omega}(z_{t+dt},n_{t+dt},\chi_{t+dt})}{dt}\right]
$$

To avoid introducing too much stochastic calculus notation, we will show that $\chi$ is a redundant state under the special case that shocks $z$ follow a multi-point Poisson jump process. The logic of the proof with other stochastic processes would be exactly the same, at the expense of more notation. In the Poisson case, we have

$$
\Gamma_z[\boldsymbol{\Omega},\Omega]=\tau(z)E_z\Big[\boldsymbol{\Omega}(\eta,n,\chi'(z,n,\chi,\eta))-\Omega(z,n,\chi)\Big]
$$

where $\tau(z)$ is the intensity at which the Poisson shocks hit, and $\eta$ is a random variable following the distribution of those shocks conditional on arrival and conditional on the initial productivity $z$.

### C.5.2 Introduce functions that update the residual $\chi$

We define the following functions given that we know how $n$ changes in each of the cases

$$
\begin{aligned}
s(z,n,\chi,\kappa(z,n,\chi)) &= (\mathcal{N}(z,n,\chi),z,s^{\chi}(z,n,\chi)) \\
d\left(z,n,\chi,i\right) &= \left(n-1,z,d^{\chi}\left(z,n,\chi,i\right)\right) \\
s\left(z,n,\chi,\left(1-\ell\left(z,n,\chi\right)\right)\circ\left(1-\left[q_{U,-i}\left(z,n,\chi\right);q_{U,i}\left(z,n,\chi\right)=1\right]\right)\right) &= (\mathcal{N}(z,n,\chi)-\tau_1(z,n,\chi),z,\tau_1^{\chi}(z,n,\chi,i)) \\
s\left(z,n,\chi,\left(1-\ell\left(z,n,\chi\right)\right)\circ\left(1-\left[q_{U,-i}\left(z,n,\chi\right);q_{U,i}\left(z,n,\chi\right)=0\right]\right)\right) &= (\mathcal{N}(z,n,\chi),z,\tau_0^{\chi}(z,n,\chi,i)) \\
s\left(z,n,\chi,\left(1-\left[\ell\left(z,n,\chi\right);\ell_i\left(z,n,\chi\right)=1\right]\right)\circ\left(1-q_U\left(z,n,\chi\right)\right)\right) &= (\mathcal{N}(z,n,\chi)-\eta_1(z,n,\chi),z,\eta_1^{\chi}(z,n,\chi,i)) \\
s\left(z,n,\chi,\left(1-\left[\ell\left(z,n,\chi\right);\ell_i\left(z,n,\chi\right)=0\right]\right)\circ\left(1-q_U\left(z,n,\chi\right)\right)\right) &= (\mathcal{N}(z,n,\chi),z,\eta_0^{\chi}(z,n,\chi,i)) \\
h_U\left(z,n,\chi\right) &= \left(n+1,z,h_U^{\chi}\left(z,n,\chi\right)\right) \\
h_E\left(z',\chi',i',z,n,\chi,n'\right) &= \left(n+1,z,h_E^{\chi}\left(z',n',\chi',i',z,n,\chi\right)\right) \\
q_E\left(z',n',\chi',i',z,n,\chi\right) &= \left(n'-1,z',q_E^{\chi}\left(n',z',\chi',i',z,n,\chi\right)\right) \\
H_n\left(z',n',\chi',i'\right) &= \frac{1}{n'}H_n\left(z',n',\chi'\right) \\
g_z\left(z,n,\chi,\eta\right) &= \Big(\eta,n,g_z^{\chi}\left(z,n,\chi,\eta\right)\Big)
\end{aligned}
$$

The above uses the function $\mathcal{N}(z,n,\chi)$, which gives the number of workers the firm retains after endogenous quits and layoffs. It solves

$$
\mathcal{N}(z,n,\chi) = \arg\max_{k\in\{0,\ldots,n\}} \Omega(k,z,\chi)+(n-k)U
$$

In addition, $\tau_1(z,n,\chi),\eta_1(z,n,\chi)\in\{0,1\}$. $\tau_1(z,n,\chi)=0$ if $\ell_i(z,n,\chi)=1$. Similarly, $\eta_1(z,n,\chi)=0$ if $q_{U,i}(z,n,\chi)=1$. Using these definitions in the Bellman equation above:

$$
\begin{aligned}
\rho\Omega\left(z,n,\chi\right) &= y\left(z,n\right)-c\left(v\left(z,n,\chi\right),z,n\right) \\
\text{Destructions} \quad & -\delta\sum_{i=1}^{n(x)}\left[\Omega\left(z,n,\chi\right)-\mathbf{\Omega}\left(n-1,z,s^{\chi}\left(z,n,\chi,i\right)\right)-U\right] \\
UE \text{ Hires} \quad & +qv\left(z,n,\chi\right)\phi\left[\Omega\left(n+1,z,h_U^{\chi}\left(z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)-U\right]\cdot\mathbb{I}_{\{(z,n,\chi)\in\mathcal{A}\}} \\
EE \text{ Hires} \quad & +qv\left(z,n,\chi\right)\left(1-\phi\right)\int_{(z,n,\chi)\in\mathcal{Q}^E(n',z',\chi',i')}\Big[\left[\mathbf{\Omega}\left(n+1,z,h_E^{\chi}\left(z',n',\chi',i',z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)\right] \\
& \qquad -\left[\Omega\left(z',n',\chi',i'\right)-\mathbf{\Omega}\left(n'-1,z',q_E^{\chi}\left(z',n',\chi',i',z,n,\chi\right)\right)\right]\Big] \\
& \qquad \cdot dH_n\left(z',n',\chi',i'\right) \\
\text{Shocks} \quad & +\tau(z)\mathbb{E}_z\Big[\mathbf{\Omega}\Big(\eta,n,g_z^{\chi}\left(z,n,\chi,\eta\right)\Big)-\Omega\left(z,n,\chi\right)\Big]
\end{aligned}
$$

and sets

$$
\begin{aligned}
\mathcal{E} &= \left\{z,n,\chi \middle| \vartheta + \mathcal{N}(z,n,\chi)\cdot U \geq \Omega(\mathcal{N}(z,n,\chi),z,s^{\chi}(z,n,\chi))\right\} \\
\mathcal{Q}^U &= \left\{ (z,n,\chi,i) \middle| \Omega(\mathcal{N}(z,n,\chi)-\tau_1(z,n,\chi),z,\tau_1^{\chi}(z,n,\chi,i)) + U \right. \\
&\qquad\qquad \left. > \Omega(\mathcal{N}(z,n,\chi),z,\tau_0^{\chi}(z,n,\chi,i)) \right\} \\
\mathcal{L} &= \left\{ (z,n,\chi,i) \middle| \Omega(\mathcal{N}(z,n,\chi)-\eta_1(z,n,\chi),z,\eta_1^{\chi}(z,n,\chi,i)) + U \right. \\
&\qquad\qquad \left. > \Omega(\mathcal{N}(z,n,\chi),z,\eta_0^{\chi}(z,n,\chi,i)) \right\} \\
\mathcal{A} &= \left\{ z,n,\chi \middle| \Omega\left(n+1,z,h_U^{\chi}(z,n,\chi)\right) - \Omega(z,n,\chi) \geq U \right\} \\
\mathcal{Q}^E\left(z',n',\chi',i'\right) &= \left\{ z,n,\chi \middle| \boldsymbol{\Omega}\left(n+1,z,h_E^{\chi}\left(z,n,\chi,z',n',\chi',i'\right)\right) - \Omega(z,n,\chi) \right. \\
&\qquad\qquad \left. \geq \Omega\left(z',n',\chi',i'\right) - \Omega\left(n'-1,z',p^{\chi}\left(z',n',\chi',i',z,n,\chi\right)\right) \right\}
\end{aligned}
$$

and the definition

$$
\mathcal{N}(z,n,\chi) = \arg\max_{k\in\{0,\ldots,n\}} \Omega(k,z,\chi) + (n-k)U
$$

and vacancy posting

$$
\begin{aligned}
\frac{c_v\left(v\left(z,n,\chi\right),z,n\right)}{q} &= \phi\left[\boldsymbol{\Omega}\left(n+1,z,h_U^{\chi}(z,n,\chi)\right) - \Omega(z,n,\chi)\right]\cdot \mathbb{I}_{\{(z,n,\chi)\in\mathcal{A}\}} \\
&+ (1-\phi)\int_{(z,n,\chi)\in\mathcal{Q}^E(z',n',\chi',i')} \Big[ \left[\boldsymbol{\Omega}\left(n+1,z,h_E^{\chi}\left(z,n,\chi,n',z',\chi',i'\right)\right) - \Omega(z,n,\chi)\right] \\
&\qquad - \left[\Omega\left(z',n',\chi',i'\right) - \boldsymbol{\Omega}\left(n'-1,z',q_E^{\chi}\left(z',n',\chi',i',z,n,\chi\right)\right)\right] \Big] \\
&\qquad \cdot\, dH_n\left(z',n',\chi',i'\right)
\end{aligned}
$$

### C.5.3 Argue that $(\chi, i)$ are a redundant state

The system above defines a functional fixed point equation. Inspection of the Bellman equation reveals that $\chi$ has no *direct* impact on the flow payoff, continuation values, or mobility sets. Its only impact is through the dependence of $\Omega$ on $\chi$. This observation implies that $\chi$ is a redundant state, and can be removed from the fixed point equation. The same argument ensures that the worker index $i$ is redundant as well.

### C.5.4 Bellman equation without $(\chi, i)$

We can re-write our Bellman equation for $(z, n) \in \mathcal{E}^c$ as:

$$
\begin{array}{rrl}
\rho\Omega(z,n) & = & y(z,n) - c(v(z,n),n) \\
\text{Destructions} & & -\delta \sum_{i=1}^{n} [\Omega(z,n) - \mathbf{\Omega}(n-1,z) - U] \\
\text{Retentions} & & +\lambda^E \sum_{i=1}^{n} \int_{(n',z') \in \mathcal{R}(z,n)} [\mathbf{\Omega}(z,n) - \Omega(z,n)] \, dH_v(x') \\
UE \text{ Hires} & & +qv(z,n)\,\phi\,[\mathbf{\Omega}(n+1,z) - \Omega(z,n) - U] \cdot \mathbb{I}_{\{(z,n)\in\mathcal{A}\}} \\
EE \text{ Hires} & & +qv(z,n)(1-\phi) \int_{(z,n)\in\mathcal{Q}^E(z',n')} \Big[ [\mathbf{\Omega}(n+1,z) - \Omega(z,n)] - [\Omega(z',n') - \mathbf{\Omega}(n'-1,z')] \Big] d\widetilde{H_n}(z',n') \\
\text{Shocks} & & +\Gamma_z[\mathbf{\Omega},\Omega](z,n)
\end{array}
$$

with the sets

$$
\mathcal{E}^c = \left\{ z,n \,\middle|\, \Omega(\mathcal{N}(z,n)) \geq \vartheta + \mathcal{N}(z,n)U \right\}
$$

$$
\mathcal{L} = \mathcal{Q}^U = \left\{ z,n \,\middle|\, \Omega(\mathcal{N}(z,n),z) - \mathbf{\Omega}(\mathcal{N}(z,n)-1,z) \leq U \right\}
$$

$$
\mathcal{A} = \left\{ z,n \,\middle|\, \mathbf{\Omega}(n+1,z) - \Omega(z,n) \geq U \right\}
$$

$$
\mathcal{Q}^E(z',n') = \left\{ z,n \,\middle|\, \mathbf{\Omega}(n+1,z) - \Omega(z,n) \geq \Omega(z',n') - \mathbf{\Omega}(n'-1,z') \right\}
$$

and the definition

$$
\mathcal{N}(z,n) = \arg\max_{k\in\{0,\ldots,n\}} \Omega(k,z) + (n-k)U
$$

and the vacancy policy function:

$$
\begin{aligned}
\frac{c_v(v(z,n),z,n)}{q} &= \phi\,[\mathbf{\Omega}(n+1,z) - \Omega(z,n)] \cdot \mathbb{I}_{\{(z,n)\in\mathcal{A}\}} \\
&\quad + (1-\phi) \int_{(z,n)\in\mathcal{Q}^E(z',n')} \Big[ [\mathbf{\Omega}(n+1,z) - \Omega(z,n)] - [\Omega(z',n') - \mathbf{\Omega}(n'-1,z')] \Big] d\widetilde{H_n}(n',z')
\end{aligned}
$$

### C.5.5 Expressing "bold" values

In this step we express "bold" values – that encode the optimal quit, layoff and exit decisions – as simple functions of non-bold values.

From the definition of the exit and quit sets $\mathcal{E}, \mathcal{Q}^U$, we can express:

$$\boldsymbol{\Omega}(z,n) \quad = \quad \max\left\{ \underbrace{\Omega(z,n)}_{\text{Operate}}, \quad \underbrace{\boldsymbol{\Omega}(n-1,z)+U}_{\text{Separate one worker and re-evaluate}}, \quad \underbrace{\vartheta+nU}_{\text{Exit}} \right\}$$

We can iterate on this equation. To see the logic, consider the first few steps.

$$\begin{aligned}
\boldsymbol{\Omega}(z,n) &= \max\left\{ \Omega(z,n)\,,\, \boldsymbol{\Omega}(n-1,z)+U\,,\, \vartheta+nU \right\} \\
&= \max\left\{ \Omega(z,n)\,,\, \max\left\{ \Omega(n-1,z)\,,\, \boldsymbol{\Omega}(n-2,z)+U\,,\, \vartheta+(n-1)U \right\}+U\,,\, \vartheta+nU \right\} \\
&= \max\left\{ \Omega(z,n)\,, \Omega(n-1,z)+U\,,\, \boldsymbol{\Omega}(n-2,z)+2U\,,\, \vartheta+(n-1)U+U\,,\, \vartheta+nU \right\} \\
&= \max\left\{ \Omega(z,n)\,, \Omega(n-1,z)+U\,,\, \boldsymbol{\Omega}(n-2,z)+2U\,,\, \vartheta+nU \right\}
\end{aligned}$$

By recursion, it is easy to see that

$$\begin{aligned}
\boldsymbol{\Omega}(z,n) &= \max\left\{ \Omega(\mathcal{N}(z,n),z)+(n-\mathcal{N}(z,n))\cdot U\,,\, \vartheta+nU \right\} \\
&= \max\left\{ \max_{k\in\{0,\ldots,n\}} \Omega(k,z)+(n-k)U\,,\, \vartheta+nU \right\}
\end{aligned}$$

where recall that

$$\mathcal{N}(z,n) = \arg\max_{k\in\{0,\ldots,n\}} \Omega(k,z)+(n-k)U$$

## C.6 Continuous workforce limit

Up to this point the economy has featured a continuum of firms, but an integer-valued workforce. We now take the continuous workforce limit by defining the 'size' of a worker—which is 1 in the integer case—and taking the limit as this approaches zero. Specifically, denote the "size" of a worker by $\Delta$, such that $n = N\Delta$ where $N$ is the old integer number of workers. Now define $\Omega^{\Delta}(z,n) := \Omega(z,n/\Delta)$, and likewise define $y^{\Delta}(z,n) := y(z,n/\Delta)$ and $c^{\Delta}(v,n,z) := c(v/\Delta,n/\Delta,z)$. We also define $b^{\Delta} := b/\Delta$ and

$\vartheta^\Delta := \vartheta/\Delta$. These imply, for example, that $\Omega(z, N) = \Omega^\Delta(z, N\Delta)$. Substituting these terms into (6) and (7), and taking the limit $\Delta \to 0$, while holding $n = N\Delta$ fixed, we would obtain a version of (8) in which all functions have the $\Delta$ super-script notation. We also specialize the productivity to a diffusion process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$.

The result is the joint value representation of section **??**: a Hamilton-Jacobi-Bellman (HJB) equation for the joint value *conditional on the firm and its workers operating*:

$$
\begin{aligned}
\rho\Omega(z,n) = \max_{v\geq 0} \quad & y(z,n) - c(v,n,z) \\
\text{Destruction} \quad & -\delta n[\Omega_n(z,n) - U] \\
UE \text{ Hire} \quad & +\phi q(\theta) v\left[\Omega_n(z,n) - U\right] \\
EE \text{ Hire} \quad & +(1-\phi) q(\theta) v \int \max\left\{\Omega_n(z,n) - \Omega_n(n',z')\,,\,0\right\} dH_n\left(z',n'\right) \\
\text{Shock} \quad & +\mu(z)\Omega_z(z,n) + \frac{\sigma(z)^2}{2}\Omega_{zz}(z,n).
\end{aligned} \tag{8}
$$

Boundary conditions for the firm and its workers operating require the state to be interior to the exit and separation boundaries:

$$
\begin{aligned}
\text{Exit boundary:} \quad & \Omega(z,n) \geq \vartheta + nU, \\
\text{Layoff boundary:} \quad & \Omega_n(z,n) \geq U
\end{aligned}
$$

Note the absence of $\boldsymbol{\Omega}$ terms. Since the value we track is that of a hiring firm subject to boundary conditions, then $\boldsymbol{\Omega} = \Omega$. This admits the simplification of 'Shock' terms we noted when discussing (2).

We proceed in three steps:

(A.5.1) Define worker size and the renormalization

(A.5.2) Take the limit as worker size goes to zero

(A.5.3) Introduce a continuous productivity process.

### C.6.1 Define worker size and the renormalization

We denote the "size" of a worker by $\Delta$. That is, we currently have an integer work-force $n \in \{1, 2, 3, \ldots\}$. We now consider $m \in \{\Delta, 2\Delta, 3\Delta, \ldots\}$. So then $n = m/\Delta$. We use this to make the following normalizations:

$$\begin{aligned}
\omega(z,m) &= \Omega\left(\frac{m}{\Delta},z\right)\\
\mathcal{Y}(z,m) &= y\left(\frac{m}{\Delta},z\right)\\
\mathcal{C}(z,m) &= c\left(\frac{v}{\Delta},\frac{m}{\Delta},z\right)
\end{aligned}$$

These definition imply

$$\begin{aligned}
\Omega(z,n) &= \omega(n\Delta,z)\\
y(z,n) &= \mathcal{Y}(n\Delta,z)\\
c(v,z,n) &= \mathcal{C}(v\Delta,n\Delta,z)
\end{aligned}$$

In addition, the value of unemployment solves

$$\rho U = b$$

Define

$$\mathcal{U} = \frac{b}{\rho\Delta} = \frac{U}{\Delta}$$

and

$$\theta = \frac{\vartheta}{\Delta}$$

Substituting these definitions into the Bellman equation, we obtain

$$\begin{aligned}
\rho\omega\left(n\Delta,z\right) = \max_{v\Delta\geq 0} \quad & \mathcal{Y}\left(n\Delta,z\right) - \mathcal{C}\left(v\Delta,n\Delta,z\right)\\
\text{Destructions} \quad & -\delta n\Delta\left[\frac{\omega\left(n\Delta,z\right)-\boldsymbol{\omega}\left(n\Delta-\Delta,z\right)}{\Delta}-\mathcal{U}\right]\\
UE \text{ Hires} \quad & +qv\Delta\phi\left[\frac{\boldsymbol{\omega}\left(n\Delta+\Delta,z\right)-\omega\left(n\Delta,z\right)}{\Delta}-\mathcal{U}\right]\cdot\mathbb{I}_{\{(n\Delta,z)\in\mathcal{A}\}}\\
EE \text{ Hires} \quad & +qv\Delta\left(1-\phi\right)\int_{(n\Delta,z)\in\mathcal{Q}^E(n'\Delta,z')}\left[\frac{\boldsymbol{\omega}\left(n\Delta+\Delta,z\right)-\omega\left(n\Delta,z\right)}{\Delta}-\frac{\omega\left(n'\Delta,z'\right)-\boldsymbol{\omega}\left(n'\Delta-\Delta,z'\right)}{\Delta}\right]d\widetilde{H_n}\left(n'\Delta,z'\right)\\
\text{Shocks} \quad & +\Gamma_z\left[\boldsymbol{\omega},\omega\right]\left(n\Delta,z\right)
\end{aligned}$$

with the set definitions

$$\mathcal{E} = \left\{ n\Delta, z \middle| \max_{k\Delta \in \{0,\ldots,n\Delta\}} \omega(k\Delta, z) + (n\Delta - k\Delta)\mathcal{U} < \theta + n\Delta\mathcal{U} \right\}$$

$$\mathcal{A} = \left\{ n\Delta, z \middle| \frac{\boldsymbol{\omega}\left(n\Delta + \Delta, z\right) - \omega\left(n\Delta, z\right)}{\Delta} \geq \mathcal{U} \right\}$$

$$\mathcal{Q}^U = \left\{ n\Delta, z \middle| \frac{\omega\left(n\Delta, z\right) - \boldsymbol{\omega}\left(n\Delta - \Delta, z\right)}{\Delta} \leq \mathcal{U} \right\}$$

$$\mathcal{Q}^E\left(n'\Delta, z'\right) = \left\{ n\Delta, z \middle| \frac{\boldsymbol{\omega}\left(n\Delta + \Delta, z\right) - \omega\left(n\Delta, z\right)}{\Delta} \geq \frac{\omega\left(n'\Delta, z'\right) - \boldsymbol{\omega}\left(n'\Delta - \Delta, z'\right)}{\Delta} \right\}$$

and the definition:

$$\boldsymbol{\omega}(n\Delta, z) \quad = \quad \max\left\{ \max_{k\Delta \in \{0,\ldots,n\Delta\}} \omega(k\Delta, z) + (n\Delta - k\Delta)\mathcal{U} \ , \ \theta + n\Delta\mathcal{U} \right\}$$

### C.6.2 Continuous limit as worker size goes to zero

Now we take the limit $\Delta \to 0$, holding $m = n\Delta$ fixed. We note $\widehat{v} = \lim_{\Delta \to 0} v\Delta$. We see derivatives appear. We denote $\omega_m(z, m) = \frac{\partial \omega}{\partial m}(z, m)$.

First, we note that the following limit obtains:

$$\boldsymbol{\omega}(z, m) \quad = \quad \max\left\{ \max_{k \in [0,m]} \omega(k, z) + (m - k)\mathcal{U} \ , \ \theta + m\Delta\mathcal{U} \right\}$$

In particular, the exit set limits to

$$\mathcal{E} = \left\{ z, m \middle| \max_{k \in [0,m]} \omega(k, z) + (m - k)\mathcal{U} < \theta + m\mathcal{U} \right\}$$

In equilibrium, the $\boldsymbol{\omega}(z, m)$ terns on the right-hand-side of the Bellman equation are the result of endogenous quits, layoffs and hires. Because our continuous time assumption has been made *before* we take the limit to a continuous workforce limit, we need only consider those changes in the workforce one at a time. Hence, for any $(z, m) \in \text{Interior}(\mathcal{E}^c \cap \mathcal{A})$, the *interior* of the continuation set, there is always $\overline{\Delta} > 0$: such that for any $\Delta \leq \overline{\Delta}$:

$$\boldsymbol{\omega}(m \pm \Delta, z) = \omega(m \pm \Delta, z)$$

Using this observation in the Bellman equation, we can obtain derivatives on the right-hand-side. We obtain, for pairs $(z, n)$ in the interior of the continuation set $(z, n) \in \text{Interior}(\mathcal{E}^c \cap \mathcal{A})$:

$$\begin{aligned}
\rho\omega(z,m) = \max_{\widehat{v}\geq 0} \quad & \mathcal{Y}(z,m) - \mathcal{C}(\widehat{v},z,m) \\
\text{Destructions} \quad & -\delta m[\omega_m(z,m) - \mathcal{U}] \\
UE \text{ Hires} \quad & +q\widehat{v}\phi\,[\omega_m(z,m) - \mathcal{U}] \cdot \mathbb{I}_{\{(z,m)\in\mathcal{A}\}} \\
EE \text{ Hires} \quad & +q\widehat{v}(1-\phi)\int_{(z,m)\in\mathcal{Q}^E(m',z')} \left[\omega_m(z,m) - \omega_m(m',z')\right] d\widetilde{H_n}(m',z') \\
\text{Shocks} \quad & +\Gamma_z[\boldsymbol{\omega},\omega](z,n)
\end{aligned}$$

with the set definitions

$$\begin{aligned}
\mathcal{E} &= \left\{ z,m \,\middle|\, \max_{k\in[0,m]} \omega(k,z) + (n-k)\mathcal{U} < \theta + m\mathcal{U} \right\} \\
\mathcal{A} &= \left\{ z,m \,\middle|\, \omega_m(z,m) \geq \mathcal{U} \right\} \\
\mathcal{Q}^U &= \left\{ z,m \,\middle|\, \omega_m(z,m) \leq \mathcal{U} \right\} = \overline{A}\text{ , the complement of }\mathcal{A} \\
\mathcal{Q}^E(z',m') &= \left\{ z,m \,\middle|\, \omega_m(z,m) - \omega_m(m',z') \geq 0 \right\}
\end{aligned}$$

and the definition

$$\boldsymbol{\omega}(z,m) = \max\left\{ \max_{k\in[0,m]} \omega(k,z) + (m-k)\mathcal{U}\ ,\ \theta + m\mathcal{U} \right\}$$

Note that now, the only place where $\boldsymbol{\omega}$ enters in the Bellman equation is the contribution of shocks. To replace it with $\omega$, we need to apply the same argument to $z$ as the one we applied to $n$. We thus need to specialize to a continuous productivity process.

### C.6.3 Continuous productivity process

We now specialize to a continuous productivity process, as this makes the formulation of the problem very economical. It allows to simplify the contribution of productivity shocks and get rid of the remaining "bold" notation. We suppose that productivity follows a diffusion process:

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$$

In this case, for any $(z, m)$ in the interior of the continuation set, productivity shocks in the interval $[t, t+dt]$ cannot move the firm towards a region in which it would endogenously separate or exit, when $dt$ is small enough. In this case, we can write the following, where we have also replaced the $\mathcal{Q}^E$ set with the max operator:

$$\begin{aligned}
\rho\omega\left(z,m\right) = \max_{v\geq 0} \quad & \mathcal{Y}\left(z,m\right) - \mathcal{C}\left(v,z,m\right) \\
\text{Destructions} \quad & -\delta m[\omega_m(z,m) - \mathcal{U}] \\
UE \text{ Hires} \quad & +qv\phi\left[\omega_m(z,m) - \mathcal{U}\right] \\
EE \text{ Hires} \quad & +qv\left(1-\phi\right)\int \max\left\{\omega_m(z,m) - \omega_m(z',m')\ ,\ 0\right\} d\widetilde{H_n}\left(m',z'\right) \\
\text{Shocks} \quad & +\mu(z)\omega_z(z,m) + \frac{\sigma(z)^2}{2}\omega_{zz}(z,m) \\
\text{s.t.} \quad & \\
\text{No Exit} \quad & \omega(z,m) \geq \theta + m\mathcal{U} \\
\text{No Separations} \quad & \omega_m(z,m) \geq \mathcal{U}
\end{aligned}$$

To make the notation more comparable, we slightly abuse notation and use the same letters as before, but now for the continuous workforce case. We obtain finally:

$$
\begin{aligned}
\rho\Omega\left(z,n\right) = \max_{v\geq 0} \quad & y\left(z,n\right) - c\left(v,z,n\right) \\
\text{Destructions} \quad & -\delta n[\Omega_n(z,n) - U] \\
UE \text{ Hires} \quad & +qv\phi\left[\Omega_n(z,n) - U\right] \\
EE \text{ Hires} \quad & +qv\left(1-\phi\right)\int \max\left[\Omega_n(z,n) - \Omega_n(z',n')\ ,\ 0\right] d\widetilde{H_n}\left(z',n'\right) \\
\text{Shocks} \quad & +\mu(z)\Omega_z(z,n) + \frac{\sigma(z)^2}{2}\Omega_{zz}(z,n) \\
\text{s.t.} \quad & \\
\text{No Exit} \quad & \Omega(z,n) \geq \vartheta + nU \\
\text{No Separations} \quad & \Omega_n(z,n) \geq U
\end{aligned}
$$

When the coalition hits $\Omega_n(z,n) = U$, it endogenous separates worker to stay on that frontier. It exits when it hits the frontier $\Omega(z,n) = \vartheta + nU$.

In addition to these "value-pasting" boundary conditions, optimality implies necessary "smooth-pasting" boundary conditions (see Stokey 2008): $\Omega_z(z,n) = 0$ if the firm actually exits at $(z,n)$ following productivity shocks, and $\Omega_n(z,n) = 0$ if the firm actually exits at $(z,n)$ following changes in size. These are necessary and sufficient for the definition of our problem (Brekke Oksendal 1990). Its general formulation terms of optimal switching between three regimes (operation, layoffs, exit) on the entire positive quadrant, can be made as a system of Hamilton-Jacobi-Bellman-Variational-Inequality (see Pham 2009), which we present here for completeness :

$$
\max \left\{ \begin{aligned}
& -\rho\Omega\left(z,n\right) + \max_{v\geq 0} -\delta n[\Omega_n(z,n) - U] + qv\phi\left[\Omega_n(z,n) - U\right] \\
& +qv\left(1-\phi\right)\int \max\left[\Omega_n(z,n) - \Omega_n(z',n')\ ,\ 0\right] d\widetilde{H_n}\left(z',n'\right) + \mu(z)\Omega_z(z,n) + \frac{\sigma(z)^2}{2}\Omega_{zz}(z,n)\ ; \\
& \vartheta + nU - \Omega(z,n)\ ;\ \max_{k\in[0,n]} \Omega(z,k) + (n-k)U - \Omega(z,n)
\end{aligned} \right\} = 0 \quad , \quad \forall (z,n) \in \mathbb{R}_+^2
$$

## D Characterization of surplus function

First define the surplus as $S(z,n) = \Omega(z,n) - nU$. Given that $\rho U = b$, this implies that $\rho S(z,n) = \rho\Omega(z,n) - nb$. We also have that $S_n(z,n) = \Omega_n(z,n) - U$. Combining these with the Bellman equation for $\Omega$:

$$\begin{aligned}
\rho S(z,n) &= \max_{v\geq 0} y(z,n) - c(v,z,n) - nb \\
&+ [q\phi v - \delta n]S_n(z,n) \\
&+ q(1-\phi)v\int_0^{S_n(z,n)}[S_n(z,n) - s]dH_n(s) \\
&+ \mu(z)S_z(z,n) + \frac{\sigma^2(z)}{2}S_{zz}(z,n)
\end{aligned}$$

where we sligtly abuse notation and use $H_n(s)$ to also denote here the employment-weighted cumulative distribution function of marginal surpluses. The value-pasting conditions become

$$\begin{aligned}
S(z,n) &\geq \vartheta \\
S_n(z,n) &\geq 0
\end{aligned}$$

We now make a number of assumptions to characterize the surplus. They are not all strictly necessary for each individual comparative static, but for convenience of exposition we present them all at the same time.

- The production function $y(z,n)$ satisfies $y_{\log z}, y_n, y_{\log z,n} > 0 > y_{nn}$.
- Productivity follows a geometric Brownian motion $\mu(z) = \mu z$ and $\sigma(z) = \sigma z$.
- Vacancy costs depend only on $v$ and are isoelastic: $c(v) = c_0 v^{1+\gamma}$.
- The surplus function is twice continuously differentiable up to the boundary of the continuation region.

We now proceed to show the comparative statics discussed in the main text.

### D.1 $S$ is increasing in $n$

The no-endogenous-separations condition $S_n \geq 0$ immediately implies that the surplus is increasing in $n$.

### D.2 $S$ is increasing in $z$

Re-write the problem in terms of $x = \log z$. Denote with a slight abuse of notation $y(x,n) = y(e^x, n)$. Then, as a function of $(x,n)$, the joint surplus solves

$$\begin{aligned}
\rho S(x,n) &= \max_{v\geq 0} y(x,n) - c(v) - nb \\
&+ [q\phi v - \delta n] S_n(x,n) \\
&+ q(1-\phi) v \mathcal{H}(S_n(x,n)) \\
&+ \left(\mu - \frac{\sigma}{2}\right) S_x(x,n) + \frac{\sigma^2}{2} S_{xx}(x,n)
\end{aligned}$$

where we integrated by parts, and denoted $\mathcal{H}(s) = \int_0^s H_n(r)dr$. Denote $\zeta(x,n) = S_x(x,n)$. Differentiate the Bellman equation w.r.t. $x$ and use the envelope theorem to obtain

$$\begin{aligned}
\rho\zeta(x,n) &= y_x(x,n) \\
&+ \left\{ \left[ q(1-\phi) H_n(S_n(x,n)) + q\phi \right] v^*(x,n) - \delta n \right\} \zeta_n(x,n) \\
&+ \mu \zeta_x(x,n) + \frac{\sigma^2}{2} \zeta_{xx}(x,n)
\end{aligned}$$

Now consider the stochastic process defined by

$$\begin{aligned}
dx_t &= \mu dt + \sigma dW_t \\
dn_t &= \left\{ \left[ q(1-\phi) H_n(S_n(x_t,n_t)) + q\phi \right] v^*(x_t,n_t) - \delta n_t \right\} dt
\end{aligned} \tag{9}$$

This correponds to the true stochastic process for productivity, but a hypothetical process for employment, that in general differes from the realized one. We can now use the Feynman-Kac formula (Pham 2009) to go back to the sequential formulation:

$$\zeta(x,n) = \mathbb{E}\left[ \int_0^T e^{-\rho t} y_x(x_t,n_t) + e^{-\rho T} \zeta(x_T,n_T) \,\middle|\, x_0 = x, n_0 = n, \{x_t,n_t\} \text{ follows (9)} \right]$$

and where $T$ is the hitting time of either the separation of exit region. By assumption, $y_x > 0$, so the contribution of the first part is always positive. On the exit region, smooth-pasting requires that $\zeta = 0$. In the interior of the separation region, $\zeta = 0$. Under our regularity assumption, we thus get $\zeta = 0$ on the layoff boundary. Thus,

$$\zeta(x,n) = \mathbb{E}\left[\int_0^T e^{-\rho t} y_x(x_t, n_t)dt \;\middle|\; x_0 = x, n_0 = n, \{x_t, n_t\} \text{ follows (9)}\right] > 0$$

which concludes the proof.

### D.3 $S$ is concave in $n$

Denote $s(z,n) = S_n(z,n)$. Differentiate the Bellman equation w.r.t. $n$ on the interior of the domain, use the envelope theorem and integrate by parts to obtain:

$$\begin{aligned}(\rho+\delta)s(z,n) &= y_n(z,n) - b \\ &+ \Big\{[q\phi + q(1-\phi)H_n(s(z,n))]v^*(z,n) - \delta n]s_n(z,n) \\ &+ \mu(z)s_z(z,n) + \frac{\sigma^2(z)}{2}s_{zz}(z,n)\end{aligned}$$

Recall that

$$(1+\gamma)c_0[v^*(z,n)]^\gamma = q\phi s(z,n) + q(1-\phi)\mathcal{H}(s(z,n))$$

In particular, differentiating w.r.t. $n$,

$$\gamma(1+\gamma)c_0[v^*(z,n)]^{\gamma-1}v_n^*(z,n) = [q\phi + q(1-\phi)H_n(s(z,n))]s_n(z,n)$$

and so

$$\gamma\frac{v_n^*(z,n)}{v^*(z,n)} = \frac{\phi + (1-\phi)H_n(s(z,n))}{\phi + (1-\phi)\overline{H}(s(z,n))}\frac{s_n(z,n)}{s(z,n)}$$

where $\overline{H}(s) = \frac{\mathcal{H}(s)}{s} \leq 1$. Now denote $\zeta(z,n) = s_n(z,n) = S_{nn}(z,n)$. Differentiate the recursion for $s$ w.r.t. $n$ to obtain

$$\begin{aligned}&\Big(\rho + 2\delta - q(1-\phi)H_n'(s(z,n)v^*(z,n)s_n(z,n) - q[\phi + (1-\phi)H_n(s(z,n))v_n^*(z,n)\Big)\zeta(z,n) \\ = \;& y_{nn}(z,n) \\ + \;& \Big\{[\lambda\phi + \lambda(1-\phi)H_n(s(z,n)]v^*(z,n) - \delta n]\zeta_n(z,n) \\ + \;& \mu(z)\zeta_z(z,n) + \frac{\sigma^2(z)}{2}\zeta_{zz}(z,n)\end{aligned}$$

Now define the "effective discount rate"

$$
\begin{aligned}
R(z,n,s_n(z,n)) &= \rho + 2\delta - q(1-\phi)H_n'(s(z,n)v^*(z,n)s_n(z,n) - q[\phi + (1-\phi)H_n(s(z,n))]v_n^*(z,n) \\
&= \rho + 2\delta - qv^*(z,n)s_n(z,n)\underbrace{\left\{(1-\phi)H_n'(s(z,n)) + \frac{\phi+(1-\phi)H_n(s(z,n))}{\gamma s(z,n)}\frac{\phi+(1-\phi)H_n(s(z,n))}{\phi+(1-\phi)\overline{H}(s(z,n))}\right\}}_{\equiv P(z,n)>0}
\end{aligned}
$$

where the second equality uses the expression for $v_n^*$ derived above. Define the stochastic process

$$
\begin{aligned}
dz_t &= \mu(z_t)dt + \sigma(z_t)dW_t \\
dn_t &= \Big\{\big[q(1-\phi)H_n(S_n(z_t,n_t)) + q\phi\big]v^*(z_t,n_t) - \delta n_t\Big\}dt \qquad (10)
\end{aligned}
$$

As before, we can use the Feynman-Kac formula to obtain

$$
\zeta(z,n) = \mathbb{E}\left[\int_0^T e^{-\int_0^t R(z_\tau,n_\tau,\zeta(z_\tau,n_\tau))d\tau} y_{nn}(z_t,n_t)dt + e^{-\int_0^T R(z_\tau,n_\tau,\zeta(z_\tau,n_\tau))d\tau T}\zeta(z_T,n_T) \;\middle|\; z_0 = z, n_0 = n, \{z_t,n_t\} \text{ follows } (10)\right]
$$

for $T$ the first hitting time of the exit/separation region. The contribution of the first term is always negative. Note that $\zeta$ enters in the effective discount rate. Inside the separation region and in the exit regions, $\zeta = 0$. We restrict attention to twice continuously differentiable functions, so $\zeta = 0$ on the exit and separation frontiers. Then

$$
\zeta(z,n) = \mathbb{E}\left[\int_0^T e^{-\int_0^t R(z_\tau,n_\tau,\zeta(z_\tau,n_\tau))d\tau} y_{nn}(z_t,n_t)dt \;\middle|\; z_0 = z, n_0 = n, \{z_t,n_t\} \text{ follows } (10)\right] < 0
$$

which concludes the proof.

### D.4 $S$ is supermodular in $(\log z, n)$

Denote again $s(x,n) = S_n(x,n)$, where $x = \log z$. Recall

$$
\begin{aligned}
(\rho+\delta)s(x,n) &= y_n(x,n) - b \\
&+ \Big\{\big[q\phi + q(1-\phi)H_n(s(x,n)\big]v^*(x,n) - \delta n\big]s_n(x,n) \\
&+ \mu s_x(x,n) + \frac{\sigma^2}{2}s_{xx}(x,n)
\end{aligned}
$$

and that

$$(1+\gamma)c_0[v^*(x,n)]^\gamma = q\phi s(x,n) + q(1-\phi)\mathcal{H}(s(x,n))$$

In particular, differentiating w.r.t. $x$,

$$\gamma \frac{v_x^*(x,n)}{v^*(x,n)} = \frac{\phi + (1-\phi)H_n(s(x,n))}{\phi + (1-\phi)\overline{H}(s(x,n))} \frac{s_x(x,n)}{s(x,n)}$$

Now denote $\zeta(x,n) = s_x(x,n) = S_{xn}(x,n)$. Differentiate the recursion for $s(x,n)$ w.r.t. $x$ to obtain

$$\begin{aligned}
&\left(\rho + \delta - q(1-\phi)H_n'(s(x,n)v^*(x,n)s_x(x,n) - q[\phi + (1-\phi)H_n(s(x,n))v_x^*(x,n)\right)\zeta(x,n) \\
= \quad & y_{nx}(x,n) \\
+ \quad & \Big\{\big[\lambda\phi + \lambda(1-\phi)H_n(s(x,n)\big]v^*(x,n) - \delta n\big]\zeta_n(x,n) \\
+ \quad & \mu\zeta_x(x,n) + \frac{\sigma^2}{2}\zeta_{xx}(x,n)
\end{aligned}$$

As before, define the "effective discount rate"

$$\begin{aligned}
R(x,n,s_x(x,n)) &= \rho + \delta - q(1-\phi)H_n'(s(x,n)v^*(x,n)s_x(x,n) - q[\phi + (1-\phi)H_n(s(x,n))]v_x^*(x,n) \\
&= \rho + \delta - qv^*(x,n)s_x(x,n)\underbrace{\left\{(1-\phi)H_n'(s(x,n)) + \frac{\phi + (1-\phi)H_n(s(x,n))}{\gamma s(x,n)}\frac{\phi + (1-\phi)H_n(s(x,n))}{\phi + (1-\phi)\overline{H}(s(x,n))}\right\}}_{\equiv P(x,n)>0}
\end{aligned}$$

where the second equality uses the expression for $v_n^*$ derived above. As before, define the stochastic process

$$\begin{aligned}
dx_t &= \mu dt + \sigma dW_t \\
dn_t &= \Big\{\big[q(1-\phi)H_n(S_n(e^{x_t}, n_t)) + q\phi\big]v^*(x_t,n_t) - \delta n_t\Big\}dt
\end{aligned} \tag{11}$$

As before, we can use the Feynman-Kac formula to obtain

$$\begin{aligned}
\zeta(x,n) = \mathbb{E}\Bigg[\int_0^T e^{-\int_0^t R(x_\tau,n_\tau,\zeta(x_\tau,n_\tau))d\tau} y_{nx}(x_t,n_t)dt + e^{-\int_0^T R(x_\tau,n_\tau,\zeta(x_\tau,n_\tau))d\tau T}\zeta(x_T,n_T) \\
\Bigg| \; x_0 = z, n_0 = n, \{x_t,n_t\} \text{ follows } (11)\Bigg]
\end{aligned}$$

for $T$ the first hitting time of the exit/separation region. The contribution of the first term is always positive. Inside the separation region and in the exit regions, $\zeta = 0$. We restrict attention to twice continuously differentiable functions, so $\zeta = 0$ on the exit and separation frontiers. Then

$$\zeta(x,n) = \mathbb{E}\left[ \int_0^T e^{-\int_0^t R(x_\tau,n_\tau,\zeta(x_\tau,n_\tau))d\tau} y_{nx}(x_t,n_t)dt \,\middle|\, x_0 = z, n_0 = n, \{x_t,n_t\} \text{ follows (11)} \right]$$

which concludes the proof.

### D.5 Net employment growth

Net employment growth in the continuation region is

$$\frac{dn_t}{dt} = q\Big[\phi + (1-\phi)H_n(S_n(z,n))\Big]v^*(z,n) - \lambda^E(1 - H_v(S_n(z,n)))n - \delta n \equiv g(z,n)$$

Using the expression above for $v^*(z,n)$:

$$\begin{aligned} g(z,n) = \; & \frac{q^{1+1/\gamma}}{[(1+\gamma)c_0]^{1/\gamma}}\Big(\phi + (1-\phi)H_n(S_n(z,n))\Big)\Big(\phi S_n(z,n) + (1-\phi)\mathcal{H}(S_n(z,n))\Big)^{1/\gamma} \\ & -\lambda^E(1 - H_v(S_n(z,n)))n - \delta n \end{aligned}$$

From the previous comparative statics on $S_n(z,n)$, it is straightforward to see that $g(z,n)$ is increasing in $\log z$ and decreasing in $n$.

# E Frictionless limits

## E.1 Setup

**Frictional problem.** Start by recalling the Bellman equation for the joint surplus in the frictional case:

$$\begin{aligned} \rho S(z,n) &= \max_v y(z,n) - nb - c(v) - \delta n S_n(z,n) \\ &+ q(\theta) v \left\{ \phi S_n + (1-\phi) \int_0^{S_n} H_n(s) ds \right\} \\ &+ (\mathbb{L}S)(z,n) \\ \text{s.t.} \quad & S(z,n) \geq 0,\ S_n(z,n) \geq 0 \end{aligned} \tag{12}$$

where $H_n$ is the employment-weighted cumulative distribution function of marginal surpluses. $\mathbb{L}$ is the differential operator that encodes the continuation value from productivity shocks. For instance, for a diffusion, $(\mathbb{L}S)(z,n) = \mu(z) S_z(z,n) + \frac{\sigma(z)^2}{2} S_{zz}(z,n)$. Recall that $\phi = \frac{u}{u+\xi(1-u)}$ is the probability that a vacancy meets an unemployed worker, and $q$ is the vacancy meeting rate.

Note that we abstracted from exogenous separations for simplicity, but endogenous separations when $S(n,z) < 0$ still occur. Denote by $\Delta$ the aggregate endogenous separation rate.

Inside the continuation region, the density function $h(z,n)$ of the distribution of firms by productivity and size is determined by the stationary KFE

$$0 = -\frac{\partial}{\partial n}\Big(h(z,n) g(z,n)\Big) + (\mathbb{L}^* h)(z,n)$$

where $\mathbb{L}^*$ is the formal adjoint of the operator $\mathbb{L}$, and $g(z,n)$ is the growth rate of employment

$$g(z,n) = q(\theta) v^*(z,n) \Big[\phi + (1-\phi) H_n(S_n(z,n))\Big] - \xi \lambda^U n \Big[1 - H_v(S_n(z,n)),\Big] \tag{13}$$

where $\lambda^U$ is the meeting rate from unemployment, and $\xi$ the relative search efficiency of the employed.

Finally, the mass of entrant firms $\mathtt{m}_0$ is determined by the free-entry condition

$$c_e = \mathbb{E}^{\text{Entry}}[\max\{S(z,n_0), 0\}] \tag{14}$$

where $n_0$ is initial employment which is a parameter, and $\mathbb{E}^{\text{Entry}}$ is the expectation operator under the productivity distribution for entrants $\Pi_0(z)$. The surplus is a function of $\mathtt{m}_0$ through the vacancy meeting rate $q(\theta)$, since $\theta$ is increasing in $\mathtt{m}_0$.

**Functional forms.** For ease of exposition, we consider isoelastic vacancy cost functions

$$c(v) = \frac{c_0}{1+\gamma} v^{1+\gamma},$$

and normalize $c_0 = 1$, but the result does not depend on the particular functional form nor on the normalization. Also, we specialize to a Cobb-Douglas matching function $\mathtt{m}(s,v) = A s^{\beta} v^{1-\beta}$, where $A$ is match efficiency, a proxy for labor market frictions. Finally, for ease of exposition, we set to zero exogenous separations to unemployment $\delta = 0$.

**Comparative statics.** We describe behavior of the economy in the limit when match efficiency $A \to \infty$. We do so for two different configurations of the economy:

1. No on-the-job-search: $\xi = 0$

2. On-the-job search: $\xi > 0$

**Notation.** We write $B \approx C$ for a first-order Taylor expansion. Denote $||S_n|| = \mathbb{E}^{Steady-state}\left[S_n^{1/\gamma}\right]^{\gamma}$, where $\mathbb{E}^{Steady-state}$ denotes the expectation under the steady-state distribution of marginal surpluses. This is also the Lebesgue $(1/\gamma)$-norm of $S_n$ under the steady-state probability measure.

### E.2 No on-the-job search

Since $\xi = 0$, $\phi = 1$. From (12), the FOC for vacancies gives

$$v^*(z,n) \quad = \quad \left(q S_n\right)^{1/\gamma}. \tag{15}$$

using this optimality condition in the value function of hiring firms:

$$\begin{aligned} \rho S(z,n) \quad &= \quad y(z,n) - nb + \frac{\gamma}{1+\gamma} \cdot q(\theta)^{\frac{1}{1+\gamma}} S_n^{\frac{1}{1+\gamma}} + (\mathbb{L}S)(z,n) \\ \text{s.t.} \qquad & S(z,n) \geq 0, \ S_n(z,n) \geq 0 \end{aligned}$$

which now only depends on $q(\theta)$ as the sole aggregate. Hence, free-entry (14) uniquely pins down $q(\theta)$ to the same value no matter what value $A$ takes. Therefore, the value function always satisfies the same Bellman equation, irrespective of $A$. Hence, throughout the state space, at any given $(n,z)$, marginal surpluses $S_n(z,n)$ remain the same as $A$ varies. Moreover, since the value $S(z,n)$ is independent from $A$, so are all the decisions by firms. As a result, the endogenous separation rate $\Delta$ always remains the same – and in particular, finite.

We now study how aggregates $\mathtt{v}, \mathtt{u}, \theta$ evolve along this limiting path. Given the matching function these determine all other equilibrium objects: $\lambda^U$, $\lambda^E$, $q$. In characterizing the limit we make use of the simple fact that both $\mathtt{m}_0$ and $\mathtt{v}$ must remain finite. If this were not the case, then infinite entry and vacancy costs would violate the economy's resource constraint.

### E.2.1 Aggregates in the limit

Integrating both sides of the FOC for vacancies under the firm distribution, and using the matching function which implies that $q = A\theta^{-\beta}$, aggregate vacancies are

$$\mathtt{v} = \mathtt{m}_0 q^{\frac{1}{\gamma}} ||S_n||^{\frac{1}{\gamma}} = \mathtt{m}_0 A^{\frac{1}{\gamma}} \theta^{-\frac{\beta}{\gamma}} ||S_n||^{\frac{1}{\gamma}}$$

Since $q$ remains constant, and $\mathtt{v}$ and $\mathtt{m}_0$ are finite in the limit, then the first equality implies that $||S_n||$ remains finite in the limit.

In the limit, the unemployment rate is $\mathtt{u} \approx \frac{\Delta}{\lambda^U}$. The matching function implies $\lambda^U = A\theta^{1-\beta}$. Combined, the unemployment rate is $\mathtt{u} \approx \Delta A^{-1}\theta^{-(1-\beta)}$. Combining these expressions with the expression for aggregate vacancies $\mathtt{v}$, tightness satisifies

$$\theta = \frac{\mathtt{v}}{\mathtt{u}} \approx \frac{\mathtt{m}_0 A^{\frac{1}{\gamma}} \theta^{-\frac{\beta}{\gamma}} ||S_n||^{\frac{1}{\gamma}}}{\Delta A \theta^{1-\beta}}$$

so that

$$\theta^{\beta\frac{1+\gamma}{\gamma}} \approx \left(\frac{\mathtt{m}_0}{\Delta}\right) ||S_n||^{\frac{1}{\gamma}} A^{\frac{1+\gamma}{\gamma}}.$$

Since $\mathtt{m}_0$, $\Delta$, and $||S_n||$ are finite, $\theta$ diverges with $A$. Therefore, $\lambda_U$ diverges as well. On the worker side, since $\lambda_U$, diverges to infinity, $\mathtt{u}$ goes to zero. On the firm side, $\mathtt{m}_0$ remains finite, but changes such that $q$ remains constant and vacancies remain finite.

### E.2.2 Invariant distribution of marginal surpluses

We now turn to the invariant distribution $h(z, n)$. After substituting optimal vacancies into (13) evaluated at $\xi = 1 - \phi = 0$, one obtains that the growth of employment in the hiring region is:

$$g(z, n) = q\Big(qS_n(z, n)\Big)^{\frac{1}{\gamma}}.$$

Since $S_n(z,n)$ remains constant throughout the state space, then employment growth in the hiring region remains constant throughout the state space. The firm loses no workers to employment because there is no on-the-job search. Since $S_n(z,n)$ and $U = b/\rho$ both stay unchanged, then the employment losses to unemployment are still unchanged. Since $S(z,n)$ is unchanged, then the exit decision is also unchanged.

Hence, the law of motion of employment is independent of $A$. Thus, the steady-state distribution $h(z,n)$ is also independent from $A$. Therefore the values of firms $S(z,n)$ are the same across the state space and the relative mass of firms at each $(z,n)$ is unchanged, despite higher but finite $\mathtt{m}_0$.

#### E.2.3 Summary

Summarizing this case: as $A \to \infty$, even though unemployment vanishes, the allocations in the search model without on-the-job search do not converge to those of a competitive firm-dynamics model. The free entry condition requires the vacancy meeting rate $q$ to remain finite and thus a non degenerate dispersion of marginal products of labor survives even in the limit as firms face the same adjustment frictions regardless of $A$. In contrast, in the competitive benchmark, marginal products of labor are equalized across firms.

### E.3 On-the-job search with a fat-tailed entry distribution

We now turn to the case in which on-the-job search remains positive at some fixed value $\xi > 0$, and thus $\phi < 1$. We follow the same logic as before, with some additional steps due to on-the-job search. To simplify algebra we abstract from exogenous job destruction, setting $\delta = 0$.

To keep the arguments manageable, we also introduce an additional assumption. We require the entry productivity distribution to have a “fat enough” tail. With decreasing returns to scale, the optimal frictionless size of the firm grows without bound as productivity becomes large. We assume that the productivity distribution of entrants is unbounded, and assume that it is fat-tailed enough that the rate at which the optimal size of a firm grows with productivity is faster rate than the decay of the productivity distribution. More precisely, the frictionless optimal size is $n^*(z) = \arg\max_n y(z,n) - bn$. We assume that the entry productivity distribution $\Pi_0(z)$ is such that

$$\lim_{z \uparrow +\infty} n^*(z)\Pi_0(z) = +\infty$$

This is satisfied for the production function $y(z,n) = zn^{\alpha}$ and the entry distribution $\Pi_0(z) \propto z^{-\zeta}$, when $\frac{1}{1-\alpha} - \zeta \geq -1$. Our empirical implementation uses these functional forms and satisfies these restrictions.[6]

[6] An alternative approach is to assume a constant arrival rate of “The Godfather” shocks that leave workers *unable to refuse*

Consider (12) written in terms of the return on a vacancy

$$\rho S(z,n) = \max_{v} y(z,n) - nb - c(v) + q(\theta)vR(S_n) + (\mathbb{L}S)(z,n)$$
$$\text{s.t.} \quad S(z,n) \geq 0,\ S_n(z,n) \geq 0$$

where

$$R(S_n) = \phi S_n + (1-\phi)\int_0^{S_n} H_n(s)ds \tag{16}$$

is the return to a vacancy. The growth of employment is

$$g(z,n) = qv^*(z,n)\Big[\phi + (1-\phi) H_n(S_n(z,n))\Big] - \xi\lambda^U n\Big[1 - H_v(S_n(z,n))\Big] \tag{17}$$

### E.3.1 Aggregates in the limit

We restrict attention to the economically meaningful case in which (1) output remains finite and strictly positive in the limit, and (2) the rate at which workers separate into unemployment remains finite in the limit. These restrictions are equivalent to a guess and verify strategy, in which we guess that (1-2) hold and then verify those conditions. The logic of our approach is then to exhibit a solution in which (1-2) hold – but in principle other cases may arise.

Consider the set of meeting rates. Because some measure $\mathtt{n}$ employed jobseekers are always present regardless of $A$, the amount of effective search effort $\mathtt{s} = \mathtt{u} + \xi\mathtt{n}$ remains finite and positive even if $\mathtt{u}$ goes to zero. By (1), vacancies also remain finite. Combined, these imply that market tightness $\theta = \mathtt{v}/\mathtt{s}$ remains finite. Since $q = A\theta^{-\beta}$ and $\lambda^U = A\theta^{-(1-\beta)}$, then both meeting rates diverge to infinity at the same rate as $A$.[7]

Consider unemployment and aggregate vacancies. (2) requires that the rate at which workers separate into unemployment is a positive constant $\Delta$ in the limit. Since $\mathtt{u} \approx \frac{\Delta}{\lambda^U}$, and $\lambda_U$ diverges, then the unemployment rate converges to zero. Since the unemployment rate converges to zero, then $\phi$ also converges to zero. Firm level and aggregate vacancies are given by

$$v = q^{\frac{1}{\gamma}} R(S_n)^{\frac{1}{\gamma}} \qquad , \qquad \mathtt{v} = \mathtt{m}_0 q^{\frac{1}{\gamma}} ||R(S_n)||^{\frac{1}{\gamma}}. \tag{18}$$

(1) implies that both aggregate vacancies $\mathtt{v}$ the mass of entering firms $\mathtt{m}_0$ remain finite. Since $\mathtt{v}$ is finite and $\mathtt{m}_0$ is finite, while $q$ diverges at the same rate as $A$, then $\gamma > 0$ requires $||R(S_n)||$ must go to zero at

any job offer. Additional details available on request.

[7] Strictly speaking, free-entry then ensures that theta is pinned down to a strictly positive value. This proof is more lengthy but does not require any additional assumptions and is available upon request.

the same rate as $A$ goes to infinity.

### E.3.2 Invariant distribution of marginal surpluses

We now show that the distribution of marginal surpluses degenerates to a single value on the support of the invariant distribution.

First, we use (18) to express firm level vacancies as a share of aggregate vacancies, where that share is determined by the firms' return on a vacancy relative to the average return:

$$v = \frac{1}{m_0}\left(\frac{R\left(S_n\right)}{||R\left(S_n\right)||}\right)^{\frac{1}{\gamma}} \mathtt{v} = \frac{1}{m_0}\left(\frac{R\left(S_n\right)}{||R\left(S_n\right)||}\right)^{\frac{1}{\gamma}}\left(\frac{\lambda^U \xi}{q}\right)$$

where the second equality uses $q = A(\mathtt{v}/\xi)^{-\beta}$, and $\lambda^U = A(\mathtt{v}/\xi)^{-(1-\beta)}$, which jointly imply that $\mathtt{v} = \lambda^U \xi / q$. Now consider the expression for growth of employment inside the continuation region (17), under the limiting case of $\phi = 0$:

$$g(z,n) \approx qvH_n\left(S_n(z,n)\right) - \xi\lambda^U n\Big[1 - H_v\left(S_n(z,n)\right)\Big]$$

Substituting in the expression for firm vacancies and collecting $\lambda^U \xi$ terms:

$$g(z,n) \approx \lambda^U \xi \left\{\frac{1}{\mathtt{m}_0}\left(\frac{R(S_n)}{||R(S_n)||}\right)^{\frac{1}{\gamma}} H_n(S_n) - n\Big[1 - H_v(S_n)\Big]\right\}.$$

Consider some $(n,z)$ that has positive mass in steady state. Since $\lambda^U$ diverges but growth must remain finite, the term in braces must be equal to zero in the limit:

$$\frac{1}{\mathtt{m}_0}\left(\frac{R(S_n)}{||R(S_n)||}\right)^{\frac{1}{\gamma}} H_n(S_n) = n\Big[1 - H_v(S_n)\Big]$$

Using this we can show that the distribution of marginal surplus converges point-wise to a degenerate limiting distribution $H_n^\infty$.

We proceed by contradiction. Suppose that $H_n$ converges to a limiting distribution $H_n^\infty$ that is non-degenerate.[8] Consider a firm at the top of the distribution, such that $1 - H_v(S_n) = 0$ . The probability that the firm loses a worker is zero, so the right-hand side is zero. However, by the supposition that $H_n$ is non-degenerate, then $R(S_n)$ converges to a non-zero value, since the firm can increase its value by poaching from workers below it on the ladder. Since there is some $R(S_n)$ that is non-zero, then $||R(S_n)||$

[8]So the probability measure of $S_n$ in the cross-section would converge in distribution to a non-degenerate limit.

also converges to a non-zero value. Therefore flows out of the firm are zero, but flows into the firm are positive. This violates the above equality, which would imply infinite growth as $\lambda^U$ diverges. This is a contradiction. Hence, in the limit $H_n^\infty$ must be degenerate, and marginal surpluses of firms converge to a common limit which we denote $S_n^*$.

We have shown that the limiting distribution $H_n^\infty$ is degenerate. This implies that the invariant distribution of employment and productivity lines up along a strip $\{z, n^*(z)\}$ where $n^*(z)$ is implicitly defined by $S_n(n^*(z), z) = S_n^*$, so is strictly increasing.[9]

### E.3.3 Unique value for $S_n^*$ on the limiting strip

We have shown that $||R(S_n)||$ and $R(S_n)$ converge to zero in the limit, yet this does not necessarily imply a particular value for $S_n^*$. Here we show that $S_n^* = 0$. We guess the following, which we verify below:

($\star$) $n^*(z) = \arg\max_n \; y(z,n) - bn$ .

From the concavity of marginal surplus and $n^*(z) > n_0$, we have

$$S(z, n_0) \leq S(z, n^*(z)) - S_n(z, n^*(z)) \times (n^*(z) - n_0)$$

In the limit $S_n(z, n^*(z)) \equiv S_n^*$ is equalized, which delivers the following upper bound to the value of entry:

$$\int S(z, n_0)\Pi_0(z)dz \leq \int S(z, n^*(z))\Pi_0(z)dz - S_n^* \int \Big(n^*(z) - n_0\Big)\Pi_0(z)dz$$

We show that $S_n^* = 0$ by contradiction. Suppose that $S_n^* > 0$. From our assumption on the entry distribution then in the limit $\int n^*(z)\Pi_0(z)dz$ is infinity. Since all other terms on the right-hand side of the above inequality are finite,[10] then a necessary condition for the above inequality to be satisfied is that $\int S(z, n_0)\Pi_0 dz < 0$, which violates the free-entry condition. Therefore it must be that $S_n^* = 0$.

Intuitively, a strictly positive marginal surplus $S_n^*$ reflects that there is an excess supply of firms in the economy relative to the supply of workers. The fat tail assumption implies that this excess supply translates into a very negative value of entry, which cannot be an equilibrium in which firms enter freely.

[9] To see that this is a strip, recall that $S(z, n)$ is such that $S_{nn} < 0$ and $S_{zn} > 0$. Therefore $S_n(n^*(z), z) = S_n^*$ implicitly defines an strictly increasing function $n^*(z)$.

[10] $\int S(z, n^*(z))\Pi_0(z)dz$ remains finite because $S(z, n^*(z))$ satisfies (12) evaluated at $(z, n^*(z))$. It can then be shown that, in the limit, $q(\theta)R(S_n^*)$ depends only on $S_n^*$ and $\theta$, but not on $A$ directly. The details of the derivation are available upon request.

#### E.3.4 Limiting value function

We now return to the limiting Bellman equation for marginal surplus. Given that $q(\theta)R(S_n^*) = 0$,[11] making the generator $\mathbb{L}$ explicit, applying the result that $n = n^*(z)$, and noting that $S_n(n,z) \geq 0$ is satisfied with equality, we have

$$\begin{aligned} \rho S(z,n^*(z)) &= y(z,n^*(z)) - n^*(z)b + \mu(z)S_z(z,n^*(z)) + \frac{\sigma(z)^2}{2}S_{zz}(z,n^*(z)) \\ \text{s.t.} \quad & S(z,n^*(z)) \geq 0 \end{aligned}$$

Our key result in the text was that the limiting economy featured a value function that depended only on $z$. However, the continuation value terms in the above value function contain *partial* derivatives with respect to $z$, not *total* derivatives. To argue that it is enough to focus on the value function evaluated on the strip, we must show that the partial derivatives approximate the total derivatives in the limit. The following shows that this is the case in the limit

$$\lim_{A\to\infty} \frac{dS(z,n^*(z))}{dz} = \lim_{A\to\infty} \left\{ \left.\frac{\partial S(z,n)}{\partial z}\right|_{n=n^*(z)} + \underbrace{\left.\frac{\partial S(z,n)}{\partial n}\right|_{n=n^*(z)}}_{\to S_n^*=0} \cdot \underbrace{\frac{dn^*(z)}{dz}}_{\text{Finite constant}} \right\} = \lim_{A\to\infty} \left.\frac{\partial S(z,n^*(z))}{\partial z}\right|_{n=n^*(z)}$$

Therefore, in the limit, exit can be described by the value function evaluated on the strip, $\overline{S}(z) := S(z,n^*(z))$ which evolves according to

$$\rho\overline{S}(z) = y(z,n^*(z)) - n^*(z)b + \mu(z)\overline{S}_z(z) + \frac{\sigma(z)^2}{2}\overline{S}_{zz}(z)$$

and an exit cut-off determined by $\overline{S}(z) = 0$.

#### E.3.5 Optimal size

We now characterize the optimal size of incumbents in the limit and verify ($\star$). We note that, if for a small period of time $dt$, the firm was away from the exit cutoff but close to its optimal size, then

$$S(z,n) \approx \Big[y(z,n) - bn\Big]dt + e^{-\rho dt}\mathbb{E}\Big[S\Big(z_{dt}, n^*(z_{dt})\Big)\Big|z_0 = z\Big]$$

[11] Details are available upon request.

because the other contributions in $n - n^*(z)$ scale with $||S_n|| = 0$. Therefore,

$$S_n(z, n^*(z)) \quad \approx \quad \Big[y_n(z, n^*(z)) - b\Big] dt$$

and so it must be that $y_n(z, n^*(z)) = b$. This confirms guess ($\star$).[12]

#### E.3.6 Summary

With on-the-job search, as $A \to +\infty$, the value function converges to the one of the Hopenhayn model. The mass of active firms converges to some finite value. Free-entry pins down the mass of firms, and converges to a condition that differs from the Hopenhayn model's. There is an additional term that stems from the value gains that entrant realize along their (very fast) growth towards their optimal size. The equilibrating variable is the limiting market tightness, that governs the size of these gains.

### E.4 On-the-job search with godfather shocks

We can relax the assumption on the entry distribution if we replace it with a "spousal" or "godfather" shock assumption. The alternative assumption is to assume that, at rate $\delta^G \xi \lambda^U$, a worker's spouse finds a job in a different location, forcing the worker to switch firms and accept any job offer. Because these relocation shocks follow from spouses' job contacts, the rate at which they affect workers scales with the overall contact rate $\lambda^U$.

We will consider the limit when the fraction of hires from spousal shocks is small, $\delta^G \to 0$. Formally, we take $A \to \infty$ first, and next $\delta^G \to 0$. The derivations follow closely those in the previous section (E.3).

The value function writes:

$$\begin{aligned}
\rho S(z,n) \quad &= \quad \max_v y(z,n) - nb - c(v) \\
&+ \quad qv \left\{ \phi S_n + (1-\phi) \int_0^{S_n} H_n(s) ds + \frac{\delta^G \xi \lambda^U (1-u)}{q\mathrm{v}} S_n \right\} - \delta^G \xi \lambda^U n S_n \\
&+ \quad \mathbb{L} S \\
\text{s.t.} \qquad & S \geq 0, \; S_n \geq 0.
\end{aligned}$$

[12] To make this argument strictly formal, differentiate the Bellman equation and represent marginal surplus as an integral with the Feynan-Kac formula as in Appendix D. The derivation details are available upon request.

Define the return to a vacancy as

$$R(S_n) = \phi S_n + (1-\phi)\int_0^{S_n} H_n(s)ds + \frac{\delta^G \xi \lambda^U (1-u)}{q\mathtt{v}} S_n$$

The growth of employment becomes

$$g(z,n) = qv^*(z,n)\left[\phi + (1-\phi)H_n(S_n(z,n)) + \frac{\delta^G \lambda^U (1-u)}{q\mathtt{v}}\right] - \xi\lambda^U n\Big[1 - H_v(S_n(z,n))\Big] - \delta^G \xi \lambda^U n$$

### E.4.1 Aggregates in the limit

The same arguments as in section E.3 apply. Because of the employed jobseekers and because vacancies remain finite by aggregate feasibility, market tightness $\theta$ remains finite as $A \to \infty$. Thus, both $q$ and $\lambda^U$ diverge with $A$.

Here too, we restrict attention to the economically meaningful case in which (1) output remains finite and strictly positive in the limit, and (2) firms remain active on average for a strictly positive time period in the limit. (1) implies that the mass of active firms $\mathtt{m}_0$ also remains finite. (2) implies that the rate at which workers separate into unemployment limits to a positive constant $\Delta$. Since $\mathtt{u} \approx \frac{\Delta}{\lambda^U}$, the unemployment rate (as well as $\phi$) converges to zero. Moreover, since

$$\mathtt{v} = \mathtt{m}_0 q^{1/\gamma} ||R\left(S_n\right)||^{1/\gamma},$$

$||R\left(S_n\right)||$ goes to zero at the same rate at which $A$ diverges.

### E.4.2 Invariant distribution of marginal surpluses

We now show that the distribution of marginal surpluses degenerates to a single value on the support of the invariant distribution.

From the KFE equation, the growth of employment inside the continuation region is:

$$g(z,n) = q\left(\phi + (1-\phi)H_n(S_n) + \frac{\delta^G \xi \theta (1-u)}{\mathtt{v}}\right)\cdot\Big(qR(S_n)\Big)^{1/\gamma} - n\lambda^U \xi\Big((1 - H_v(S_n(z,n)) + \delta^G\Big)$$

Using the expression for $\mathtt{v}$ and $u$ in the limit, we again obtain that

$$\frac{q^{\frac{1+\gamma}{\gamma}}}{\lambda^U} \approx \xi \mathtt{m}_0^{-1} ||R(S_n)||^{-\frac{1}{\gamma}}.$$

Using this expression and recalling that in the limit $\phi = 0$ and that $\mathrm{v} = \xi\theta$,

$$g(z,n) \approx \lambda^U \cdot \left\{\xi \mathtt{m}_0^{-1}\Big(H_n(S_n)+\delta^G\Big)\left(\frac{R(S_n)}{||R(S_n)||}\right)^{1/\gamma} - \xi n\Big(1-H_v(S_n)+\delta^G\Big)\right\}.$$

Since $\lambda^U$ diverges, at the points in the state space that have positive employment in steady state the bracket must vanish in the limit, i.e.

$$\mathtt{m}_0^{-1}\Big(H_n(S_n)+\delta^G\Big)\left(\frac{R(S_n)}{||R(S_n)||}\right)^{1/\gamma} = n\Big(1-H_v(S_n)+\delta^G\Big)$$

Now consider the limit in which the spousal shocks are small, $\delta^G \to 0$. Then we obtain

$$\mathtt{m}_0^{-1}H_n(S_n)\left(\frac{R(S_n)}{||R(S_n)||}\right)^{1/\gamma} = n\Big(1-H_v(S_n)\Big)$$

Now suppose for a contradiction that $H_n$, the cumulative distribution function of $S_n$, converges pointwise to a non-degenerate limiting distribution $H_n^\infty$.[13] Then $||R(S_n)||$ converges to a non-zero value. Consider a firm at the top of the distribution, such that $1-H_v(S_n)=0$. For such firm $R(S_n)>0$. With a non-degenerate distribution, the left-hand-side cannot be zero in the limit, a contradiction.

We have again shown that the invariant distribution of marginal surpluses concentrates on a strip $\{z, n^*(z)\}_z$ where the limiting marginal surplus is equalized.[14] We denote $S_n^*$ its common limit. In addition, to a second order, $S_n(z,n^*(z)) = S_n^* + \mathcal{O}(\delta^G)$

### E.4.3 Unique value for $S_n^*$ on the limiting strip, and finite value for labor market tightness $\theta$

To pin down labor market tightness, we return to the maximized value. Given that firms jump to their optimal size, we can evaluate $R(S_n)$ at $S_n^*$ for each term that corresponds to a jump. We arrive at the following expression[15] up to a second order in $\delta^G$,

$$\begin{aligned}\rho S(z,n) &= y(z,n) - nb \\ &+ \frac{\gamma}{1+\gamma}\left\{\frac{\kappa_0 S_n + \kappa_0 I_0 S_n^* + \xi\delta^G\Lambda(\theta)}{\theta}\right\}^{\frac{1+\gamma}{\gamma}} - \xi\delta^G\Lambda(\theta)n \\ &+ \mathbb{L}S\end{aligned}$$

where $\kappa_0 = \frac{\Delta}{\xi}$, $I_0 \geq 0$ and $\Lambda(\theta) = \lambda^U S_n^*$.

[13] So the probability measure of $S_n$ in the cross-section would converge in distribution to a non-degenerate limit.
[14] We can argue that it is such a strip using the concavity properties shown in Appendix D.
[15] The details of the derivation are available upon request.

For the value of entry to remain finite, it must be that $\Lambda(\theta)$ remains finite as $A \to +\infty$, for any $\delta^G > 0$. Since $\Lambda(\theta) = \lambda^U S_n^*$ and $\lambda^U \to +\infty$, $S_n^* \to 0$ up to a second order in $\delta^G$.

Using free-entry again, we can now draw two additional conclusions. First, $\Lambda\delta^G$ converges to a constant as $\delta^G \to 0$. That constant may be positive, but may also be zero. Denote that constant

$$w - b \equiv \lim_{\delta^G \downarrow 0} \xi \Lambda \delta^G \geq 0$$

Define also

$$C_f(w, \theta) = -\lim_{\delta^G \downarrow 0} \frac{\kappa_0 \gamma}{1+\gamma} \left\{ \frac{w-b}{\theta} \right\}^{\frac{1+\gamma}{\gamma}} .$$

Substituting these definitions into the Bellman equation, we obtain

$$\rho S(z,n) \quad = \quad y(z,n) - nw - C_f + \mathbb{L} S \, .$$

### E.4.4 Limiting value function

The value function evaluated on the strip $\{z, n^*(z)\}_z$ – where $S_n = 0$ in the limit – converges to the solution of

$$\begin{aligned} \rho S(z, n^*(z)) \quad &\approx \quad y(z, n^*(z)) - n^*(z) w - C_f \\ &+ \quad \mu(z) \frac{\partial S}{\partial z}(z, n^*(z)) + \frac{\sigma^2(z)}{2} \frac{\partial^2 S}{\partial z^2}(z, n^*(z)) \\ & \qquad S(z, n^*(z)) \geq 0 \end{aligned}$$

where recall that the continuation term vanishes for *incumbent* firms. We have made explicit the generator $\mathbb{L}$. Recall that the candidate limiting competitive economy has a value function that depends only on $z$. However, the continuation value terms here contain *partial* derivatives w.r.t. $z$, not *total* derivatives. To argue that it is enough to focus on the value function evaluated on the strip, we must show that the partial derivatives approximate the total derivatives in the limit. To see this, compute

$$\frac{dS(z, n^*(z))}{dz} = \frac{\partial S}{\partial z}(z, n^*(z)) + \frac{\partial S}{\partial n}(z, n^*(z)) \cdot \frac{dn^*(z)}{dz}$$

But now recall that $\frac{\partial S}{\partial n}(z, n^*(z)) \to 0$. In addition, $\frac{dn^*(z)}{dz}$ stays finite by definition of $n^*(z)$. Therefore, as $A \to \infty$,

$$\frac{dS(z, n^*(z))}{dz} \approx \frac{\partial S}{\partial z}(z, n^*(z))$$

Therefore, in the limit, the exit behavior of an existing firm can be described by the firm's realized value:

$$\begin{aligned}\overline{S}(z) &\equiv S(z, n^*(z)) \\ \rho\overline{S}(z) &= y(z, n^*(z)) - n^*(z)w - C_f + \mathbb{L}\overline{S}(z) \\ &\overline{S}(z) \geq 0\end{aligned}$$

### E.4.5 Optimal size

We now characterize the optimal size of incumbents in the limit. We note that, for a small period of time $dt$, away from the exit cutoff and close to the optimal size,

$$S(z, n) \approx \Big[y(z, n) - wn\Big]dt + e^{-\rho dt}\mathbb{E}\Big[S(z_{dt}, n^*(z_{dt})|z_0 = z\Big]$$

because the other contributions in $n - n^*(z)$ scale with $||S_n|| = 0$. Therefore,

$$S_n(z, n^*(z)) \approx \Big[y_n(z, n^*(z)) - w\Big]dt$$

and so it must be that $y_n(z, n^*(z)) = w$.

### E.4.6 Summary

With on-the-job search, as $A \to +\infty$, the value function converges to one of the Hopenhayn model, and a wage $w \geq b$. The mass of active firms converges to some finite value. Free-entry pins down the mass of firms, and converges to a condition that differs from the Hopenhayn model's. There is an additional term that stems from the value gains that entrant realize along their (very fast) growth towards their optimal size.

The presence of godfather shocks requires coalitions to pay the limiting marginal surplus $S_n^*$ very frequently as they hire (or lose) many workers each instant. This extra value spending reduces the optimal size of a coalition. As $S_n^*$ goes to zero in the limit, but coalitions must pay this small cost more and more frequently, the total cost per worker stays finite. As a result, the frictionless limit resembles the Hopenhayn economy, but with an endogenous cost of labor that reflects marginal surplus spending by coalitions. This endogenous cost of labor plays the role of a limiting wage. Similarly, part of that marginal surplus spending results in an endogenous contribution to the fixed cost.

# F Algorithm

Recall that the problem facing the coalition is to optimally separate, exit and post vacancies such as to maximize the joint coalition value,

$$
\begin{aligned}
\rho\Omega(z,n) &= \max_{v\geq 0} \quad y(z,n) - \delta n\left[\Omega_n(z,n) - U\right] \\
&+ q\left[\phi\left[\Omega_n(z,n) - U\right] + (1-\phi)\int \max\left\{\Omega_n(n,z) - \Omega_n\left(n',z'\right),0\right\} dH_n\left(n',z'\right)\right]v \\
&- c(v,n) \\
&+ \mu(z)\Omega_z(z,n) + \frac{\sigma^2(z)}{2}\Omega_{zz}(z,n) \\
s.t. & \\
\Omega(z,n) &\geq nU + \vartheta \\
\Omega_n(z,n) &\geq U
\end{aligned}
\tag{19}
$$

We can rewrite the term under the integral sign in (19) to integrate directly over $\Omega_n' = \Omega_n\left(n',z'\right)$

$$\int_U^{\Omega_n(n,z)} \left[\Omega_n(n,z) - U\right] - \left[\Omega_n' - U\right]\ dH_n\left(\Omega_n'\right)$$

where we used the fact that the lower bound of the support must be $U$, since $\Omega_n' \geq U$ and the upper bound is given by the fact that a firm only hires if $\Omega_n' \leq \Omega_n(n,z)$, and we added and subtracted $U$ in the integrand. Since $\Omega_n' \in [U, \Omega_n(n,z)]$ implies $\Omega_n' - U \in [0, \Omega_n(n,z) - U]$, we can integrate over $\Omega_n' - U$ and adjust the bounds

$$\int_0^{\Omega_n(n,z)-U} \left[\Omega_n(n,z) - U\right] - \left[\Omega_n' - U\right]\ dH_n\left(\Omega_n' - U\right)$$

We can hence restate the problem (19) as

$$
\begin{aligned}
\rho\Omega(z,n) &= \max_{v\geq 0} \quad y(z,n) - \delta n\left[\Omega_n(z,n) - U\right] \\
&+ \quad q\left[\phi\left[\Omega_n(z,n) - U\right] + (1-\phi)\int_0^{\Omega_n(n,z)-U} \left[\Omega_n(n,z) - U\right] - \left[\Omega_n' - U\right]\, dH_n\left(\Omega_n' - U\right)\right] v \\
&- \quad c(v,n) \\
&+ \quad \mu(z)\Omega_z(z,n) + \frac{\sigma^2(z)}{2}\Omega_{zz}(z,n) \\
s.t. & \\
\Omega(z,n) &\geq nU + \vartheta \\
\Omega_n(z,n) &\geq U
\end{aligned} \tag{20}
$$

Let $S(z,n)$ be the *surplus* of the coalition, $S(z,n) = \Omega(z,n) - nU$. Note that $S_z(z,n) = \Omega_z(z,n)$, $S_{zz}(z,n) = \Omega_{zz}(z,n)$ and $S_n(z,n) = \Omega_n(z,n) - U$. Substituting this in problem (20),

$$
\begin{aligned}
\rho S(z,n) &= \max_{v\geq 0} \quad y(z,n) - \delta n S_n(z,n) \\
&+ \quad q\left[\phi S_n(z,n) + (1-\phi)\int_0^{S_n(z,n)} S_n(z,n) - S_n'\, dH_n\left(S_n'\right)\right] v \\
&- \quad c(v,n) \\
&+ \quad \mu(z) S_z(z,n) + \frac{\sigma^2(z)}{2} S_{zz}(z,n) - \rho n U \\
s.t. & \\
S(z,n) &\geq \vartheta \\
S_n(z,n) &\geq 0
\end{aligned}
$$

Integrate by parts the expected value of a vacancy conditional on meeting an employed worker

$$
\begin{aligned}
\int_0^{S_n(z,n)} \left[S_n(z,n) - S_n'\right] dH_n\left(S_n'\right) &= \left[S_n(z,n) - S_n'\right] H_n\left(S_n'\right)\Big|_0^{S_n(z,n)} + \int_0^{S_n(z,n)} H_n\left(S_n'\right) dS_n' \\
&= \left[S_n(z,n) - S_n(z,n)\right] H_n\left(S_n(z,n)\right) \\
&- \left[S_n(z,n) - 0\right] H_n(0) + \int_0^{S_n(z,n)} H_n\left(S_n'\right) dS_n'
\end{aligned}
$$

The second term on the second line is equal to zero since the constraint on the firms' problem $S_n(z,n) \geq 0$

implies that the distribution of marginal surpluses at other firms must also be zero at zero

$$\int_0^{S_n(z,n)} \left[S_n(z,n) - S'_n\right] dH_n\left(S'_n\right) = \int_0^{S_n(z,n)} H_n\left(S'_n\right) dS'_n$$

Recall that we defined $\mathcal{H}(x)$ as the integral of the cdf $H_n$: $\mathcal{H}(x) = \int_0^x H_n(\mathtt{u})\, du$

$$\int_0^{S_n(z,n)} \left[S_n(z,n) - S'_n\right] dH_n\left(S'_n\right) = \mathcal{H}\left(S_n(z,n)\right)$$

Substituting this into the Bellman equation

$$\begin{aligned}
\rho S(z,n) &= \max_{v\geq 0} \quad y(z,n) - \delta n S_n(z,n) \\
&+ \quad q\left[\phi S_n(z,n) + (1-\phi)\,\mathcal{H}\left(S_n(z,n)\right)\right] v \\
&- \quad c(v,n) \\
&+ \quad \mu(z) S_z(z,n) + \frac{\sigma^2(z)}{2} S_{zz}(z,n) - \rho n U \\
s.t.& \\
S(z,n) &\geq 0 \\
S_n(z,n) &\geq 0
\end{aligned}$$

We assume that the vacancy cost satisfies $c(v,n) = \bar{c}\left(\frac{v}{n}\right) v$, where $\bar{c}$ is iso-elastic with elasticity $\gamma$. Define the function $\mathcal{H}(x)$ by $\mathcal{H}(x) = q\left[\phi x + (1-\phi)\,\mathcal{H}(x)\right]$. Substituting this into problem (20)

$$\begin{aligned}
\rho S(z,n) &= \max_{v\geq 0} \quad y(z,n) - \delta n S_n(z,n) \\
&+ \quad \mathcal{H}\left(S_n(z,n)\right) v - c(v,n) \\
&+ \quad \mu(z) S_z(z,n) + \frac{\sigma^2(z)}{2} S_{zz}(z,n) - \rho n U
\end{aligned}$$

Since $\bar{c}(v/n)$ is iso-elastic in $(v/n)$, $c_v(v,n) = (\gamma+1)\,\bar{c}\left(\frac{v}{n}\right)$.[16] Along with the first order condition $c_v(v,n) = \mathcal{H}\left(S_n(z,n)\right)$, this implies

$$c(v,n) = \bar{c}\left(\frac{v}{n}\right) v = \frac{1}{\gamma+1} c_v(v,n)\, v = \frac{1}{\gamma+1}\mathcal{H}\left(S_n(z,n)\right) v$$

---

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$$c_v = \bar{c}'\frac{v}{n} + \bar{c} = \left(\frac{\bar{c}'}{\bar{c}}\frac{v}{n} + 1\right)\bar{c} = (\gamma+1)\,\bar{c}$$

Therefore the total value of vacancy posting is

$$\mathcal{H}\left(S_n\left(z,n\right)\right)v - c\left(v,n\right) = \frac{\gamma}{\gamma+1}\mathcal{H}\left(S_n\left(z,n\right)\right)v$$
$$\mathcal{H}\left(S_n\left(z,n\right)\right)v - c\left(v,n\right) = \frac{\gamma}{\gamma+1}\mathcal{H}\left(S_n\left(z,n\right)\right)\left(\frac{v}{n}\right)n$$

Letting $\bar{c}\left(\frac{v}{n}\right) = \frac{\kappa}{1+\gamma}\left(\frac{v}{n}\right)^{\gamma}$ and using $\bar{c}\left(\frac{v}{n}\right) = \frac{1}{\gamma+1}\mathcal{H}\left(S_n\left(z,n\right)\right)$ then

$$\frac{v}{n} = \kappa^{-1/\gamma}\mathcal{H}\left(S_n\left(z,n\right)\right)^{\frac{1}{\gamma}}$$

and

$$\mathcal{H}\left(S_n\left(z,n\right)\right)v - c\left(v,n\right) = \frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\mathcal{H}\left(S_n\left(z,n\right)\right)^{\frac{\gamma+1}{\gamma}}n$$

Substituting this into the Bellman equation

$$\begin{aligned}\rho S\left(z,n\right) &= y\left(z,n\right) - \delta n S_n\left(z,n\right) \\ &+ \frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\mathcal{H}\left(S_n\left(z,n\right)\right)^{\frac{\gamma+1}{\gamma}}n \\ &+ \mu(z)S_z(z,n) + \frac{\sigma^2(z)}{2}S_{zz}(z,n) - \rho n U\end{aligned}$$

Collecting terms and recognizing that $\rho U = b$,

$$\begin{aligned}\rho S\left(z,n\right) &= y\left(z,n\right) - bn \\ &+ \left[\frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\frac{\mathcal{H}\left(S_n\left(z,n\right)\right)^{\frac{\gamma+1}{\gamma}}}{S_n\left(z,n\right)} - \delta\right]S_n\left(z,n\right)n \\ &+ \mu(z)S_z(z,n) + \frac{\sigma^2(z)}{2}S_{zz}(z,n)\end{aligned} \tag{21}$$

subject to

$$\begin{aligned}S(z,n) &\geq 0 \\ S_n(z,n) &\geq 0 \\ \mathcal{H}\left(S_n\left(z,n\right)\right) &= q\left[\phi S_n\left(z,n\right) + \left(1-\phi\right)\mathcal{H}\left(S_n\left(z,n\right)\right)\right] \\ \mathcal{H}\left(S_n\left(z,n\right)\right) &= \int_0^{S_n(z,n)} H_n\left(s\right)ds\end{aligned}$$

## F.1 Algorithm

The algorithm consists of three steps, implemented in MATLAB called from master file `MAIN.m`.

**Step 0: Construct an initial guess.** Start by constructing a $n_z \times n_n$ grid for log productivity and log size. Let $\boldsymbol{\pi} = y(z,n) - bn$ denote the stacked $(n_z * n_n) \times 1$ vector of flow payoffs on this grid. Guess an initial surplus $\boldsymbol{S}^0$ on this grid (a $(n_z * n_n) \times 1$ column vector); a distribution of firms over productivity and size $\boldsymbol{h}^0$ (a $(n_z * n_n) \times 1$ column vector); aggregate finding rates $q^0$ and $\lambda^0$; and an efficiency-weighted share of unemployed searchers, $\theta^0$. Construct marginal surplus. Construct exit regions, separation regions and the vacancy policy. File `InitialGuess.m` does this.

**Step I: Iterate to convergence the coalition's problem for given aggregate states.** For $t \geq 1$, given $q^{t-1}$, $\theta^{t-1}$, $h^{t-1}$ and $\boldsymbol{S}^{t-1}$, solve the coalition's problem to update the coalition value to $\boldsymbol{S}^t$. The solution to the coalition's surplus function is obtained in an inner iteration $\tau$. Denote by $\boldsymbol{S}^{t,\tau}$ the surplus in outer iteration $t$ during inner iteration $\tau$, initiated with $\boldsymbol{S}^{t,0} = \boldsymbol{S}^t$; $\boldsymbol{T}_n(z,n)$ a $(n_z * n_n) \times (n_z * n_n)$ matrix such that $\boldsymbol{S}_n^{t,\tau} = \boldsymbol{T}_n(z,n)\boldsymbol{S}^{t,\tau}$, where $\boldsymbol{S}_n^{t,\tau}$ is the stacked $(n_z * n_n) \times 1$ vector of derivatives of $S$ w.r.t. $n$ during outer iteration $t$ and inner iteration $\tau$; $\boldsymbol{T}_z$ a $(n_z * n_n) \times (n_z * n_n)$ matrix such that $\boldsymbol{S}_z^{t,\tau} = \boldsymbol{T}_z\boldsymbol{S}^{t,\tau}$, where $\boldsymbol{S}_z^{t,\tau}$ is the stacked $(n_z * n_n) \times 1$ vector of derivatives of $S$ w.r.t. $z$ during outer iteration $t$ and inner iteration $\tau$; and $\boldsymbol{T}_{zz}$ a $(n_z * n_n) \times (n_z * n_n)$ matrix such that $\boldsymbol{S}_{zz}^{t,\tau} = \boldsymbol{T}_{zz}\boldsymbol{S}^{t,\tau}$, where $\boldsymbol{S}_{zz}^{t,\tau}$ is the stacked $(n_z * n_n) \times 1$ vector of second derivatives of $S$ w.r.t. $z$ during outer iteration $t$ and inner iteration $\tau$. Note that the matrix $\boldsymbol{T}_n(z,n)$ depends on $(z,n)$ in the sense that the approximation is done either forward or backward depending on the endogenous drift for $n$ at $(z,n)$ (note that the drift of and innovations to $z$ are independent of $(z,n)$). Within each outer iteration $t$, we iteratively update $\boldsymbol{S}^{t-1,\tau}$ for $\tau \geq 1$ following equation (21) based on

$$\left[\left(\rho + \frac{1}{\Delta}\right)\mathbb{1} - \left[\frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\frac{\mathcal{H}\left(\boldsymbol{S}_n^{t-1,\tau-1}\right)^{\frac{\gamma+1}{\gamma}}}{\boldsymbol{S}_n^{t-1,\tau-1}} - \delta\mathbb{1}\right] .* \ T_n\left(z,n\right) - \mu T_z - \frac{\sigma^2}{2}T_{zz}\right]\boldsymbol{S}^{t-1,\tau} \quad = \quad \boldsymbol{\pi} + \frac{1}{\Delta}\boldsymbol{S}^{t-1,\tau-1}$$

where $\Delta$ is the step size, $.*$ denotes the element-by-element product, and $\mathcal{H}\left(\boldsymbol{S}_n^{t-1,\tau-1}\right)^{\frac{\gamma+1}{\gamma}} / \boldsymbol{S}_n^{t-1,\tau-1}$ is a $(n_z * n_n) \times (n_z * n_n)$ matrix constructed using the previous iteration's derivative of $S$ stacked $(n_z * n_n)$ times in the column dimension. The step size cannot be too large for the problem to converge. These iterations are performed by iterating on $\tau$ until convergence by file `IndividualBehavior.m`, and the solution is assigned as the updated $\boldsymbol{S}^t$. We also obtain from the converged solution the updated separation, exit and a vacancy policies.

**Step II: Iterate to convergence the aggregate states for given individual behavior.** Given updated individual behavior in outer iteration $t$, obtain through iteration in an inner loop $\tau$ the distribution of firms $\boldsymbol{h}^t$, the aggregate meeting rates $q^t$ and $\lambda^t$, the share of unemployed searchers $\theta^t$, the distribution of workers over marginal surplus $\boldsymbol{H}_n^t$, and the distribution of vacancies over marginal surplus $\boldsymbol{H}_v^t$. File `AggregateBehavior.m` proceeds to do this in four steps.

Initiate each aggregate object with the previous outer iteration solution, $\boldsymbol{x}^{t-1,0} = \boldsymbol{x}^{t-1}$. Then:

*Step II-a.* Update the distribution of workers over marginal surplus to $\boldsymbol{H}_n^{t-1,\tau}$ given a distribution of firms $\boldsymbol{h}^{t-1,\tau-1}$ and marginal surplus $\boldsymbol{S}_n^t$, where the latter was obtained in **Step I** above. This is done by file `CdfG.m`.

*Step II-b.* Update the distribution of vacancies over marginal surplus $\boldsymbol{H}_v^{t-1,\tau}$ given a distribution of firms $\boldsymbol{h}^{t-1,\tau-1}$, the vacancy policy $\boldsymbol{v}^t$ and the ranking of firms in marginal surplus space. This is done by file `CdfF.m`.

*Step II-c.* Update the finding rates $q^{t-1,\tau}$, $\lambda^{t-1,\tau}$ and $\theta^{t-1,\tau}$ that is consistent with the vacancy policy $\boldsymbol{v}^t$ and the distribution of firms $\boldsymbol{h}^{t-1,\tau-1}$. This is done by file `HazardRates.m`.

*Step II-d.* Given $\boldsymbol{H}_n^{t-1,\tau}$, $\boldsymbol{H}_v^{t-1,\tau}$, $q^{t-1,\tau}$, $\lambda^{t-1,\tau}$ and $\theta^{t-1,\tau}$, update the distribution of firms $\boldsymbol{h}^{t-1,\tau}$ following the Kolmogorov forward equation in steady-state. This is executed by file `Distribution.m`.

Iterate over the four sub-steps *Step II-a*–*Step II-d* until convergence and assign the updated aggregate states $q^t$, $\lambda^t$, $\theta^t$ and $\boldsymbol{h}^t$. We subsequently return to step **Step I** and iterate on step **Step I–Step II** until both the surplus function and the aggregate states have converged.

## F.2 Estimation

The criterion function that we minimize is highly-dimensional and potentially has many local minima. Furthermore, the equilibrium does not exist for some regions of the parameter space. For example, if the drift in productivity is not sufficiently negative, there is no ergodic distribution for productivity. For these reasons, using a sequential hill-climbing optimizer that updates its initial guess sequentially through a gradient-based method is prohibitive. Our solution is to use an algorithm that we can easily parallelize, that efficiently explores the parameter space, and for which we can ignore cases with no equilibrium.

We set up a hyper-cube in the parameter space and then initialize a Sobol sequence to explore it. A Sobol sequence is a quasi-random low-discrepancy sequence that maintains a maximum dispersion in each dimension and far outperforms standard random number generators. We then partition the sequence and submit each partition to a separate CPU on a high performance computer (HPC). From each evaluation of the parameter hyper-cube, we save the vector of model moments. We then collect them, splice them all together, and choose the one that minimizes the criterion function. Starting with wide bounds on the parameters, we run this procedure a number of times, shrinking the hyper-cube step by step until we achieve the global minimum.

Compared to standard optimizers, this procedure has the advantage that, as a byproduct of the estimation, we can learn a lot about model identification. From an optimizer one may retrieve the moments of the model only along the path of the parameter vector chosen by the algorithm. In our case, we retrieve tens of thousands of evaluations, knowing that the low-discrepancy property of the Sobol sequence implies that for an interval of any one parameter, the remaining parameters are drawn uniformly. Plotting each single moment against parameters therefore shows the effect of a parameter on a certain moment, conditional on local draws of all other parameters.